ON k-GONAL LOCI IN SEVERI VARIETIES ON GENERAL $K3$ SURFACES AND RATIONAL CURVES ON HYPERKÄHLER MANIFOLDS (FIRST VERSION, SUPERSEDED BY ARXIV:1204.4838)

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ABSTRACT. In this paper we study the gonality of the normalizations of curves in the linear system $|H|$ of a general primitively polarized $K3$ surface (S, H) of genus p. We prove two main results. First we give a necessary condition on p, g, k for the existence of a curve in $|H|$ with geometric genus g whose normalization has a g^1_k. Secondly we prove that for p even and all numerical cases compatible with the above necessary condition, there is a family of *nodal* curves $|H|$ with the given g, k and of dimension equal to the *expected dimension* $\min\{2(k-1), g\}$. For odd p the result is only slightly less sharp. Relations with the Mori cone of the hyperkähler manifold $\mathrm{Hilb}^k(S)$ and with conjectures by Hassett-Tschinkel and by Huybrechts-Sawon are discussed.

This version is superseded by the new submission arXiv:1204.4838 where Theorem 0.1 is improved to include the missing case and the degeneration argument in its proof is made considerably simpler. Since the degeneration argument in the present version is of a different type, and may be useful for other purposes, we choose to keep this submission as well.

CONTENTS

INTRODUCTION

Let (S, H) be a general primitively polarized $K3$ surface of genus $p \geq 2$, i.e. $\Omega^2_S \simeq \mathcal{O}_S$, $h^1(\mathcal{O}_S) = 0$ and H is a globally generated, indivisible, divisor with $H^2 = 2p - 2$. In this paper we study the gonality of the normalization of curves, specifically of *nodal* curves, in the linear system $|H|$.

Let $V_{|H|,\delta} \subseteq |H|$ be the *Severi variety* of curves with $\delta \leq p$ nodes. It is a classical result that $V_{|H|,\delta}$ is a nonempty, locally closed, smooth variety of dimension $g = p - \delta$, which is the geometric genus of the curves in $V_{|H|,\delta}$. The moduli morphism $V_{|H|,\delta} \to \mathcal{M}_g$ is finite to its image, which has dimension g (see Proposition 1.2 below).

We consider $V^k_{|H|,\delta} \subseteq V_{|H|,\delta}$ the subvariety of curves whose normalizations carry a g^1_k. By Brill-Noether theory, if $g \leq 2(k-1)$, then $V^k_{|H|,\delta} = V_{|H|,\delta}$ so the interesting case is $g > 2(k-1)$. Then a count of parameters, carried out in §1.4, suggests that the expected dimension of $V^k_{|H|,\delta}$ is $2(k-1)$. In any event, if nonempty, $V^k_{|H|,\delta}$ has dimension at least $2(k-1)$ (see Proposition 1.4).

Our first main result is Theorem 3.1, which yields a necessary condition for the normalization of a curve $C \in |H|$, of geometric genus g (with any type of singularities), on a primitively polarized $K3$ surface (S,H) of genus p with $\mathrm{Pic}(S) \simeq \mathbb{Z}[H]$ to possess a g^r_d, namely that

$$\rho(p, \alpha r, \alpha d + \delta) \geq 0, \quad \text{where} \;\; \alpha := \left\lfloor \frac{gr + (d-r)(r-1)}{2r(d-r)} \right\rfloor,$$

and ρ is the usual *Brill-Noether number.* In particular, setting $\delta := p - g$, the necessary condition to possess a g^1_k is

$$\delta \geq \alpha\Big(p - \delta - (k-1)(\alpha+1)\Big) \;\; \text{where} \;\; \alpha := \left\lfloor \frac{p-\delta}{2(k-1)} \right\rfloor. \tag{1}$$

Theorem 3.1 is a strong improvement of [FKP2, Thm. 1.4] and its proof is based on the vector bundle approach à la Lazarsfeld [La].

Our second main result deals with nonemptiness and dimension of $V^k_{|H|,\delta}$:

Theorem 0.1. *Let (S,H) be a general primitively polarized $K3$ surface of genus $p \geq 2$ and let δ and k be integers satisfying $0 \leq \delta \leq p$ and $k \geq 2$. Set $\epsilon := p - 2\lfloor \frac{p}{2} \rfloor$ and α as in (1). If*

$$\delta \geq \alpha\Big(p - \delta - (k-1)(\alpha+1)\Big) + \epsilon, \tag{2}$$

then $V^k_{|H|,\delta}$ has an irreducible component of dimension $\min\{2(k-1), p-\delta\}$ *and the general element therein is a curve with only non-neutral nodes with respect to the g^1_k on its normalization, and this g^1_k has simple ramification.*

One has $\epsilon = 0$ if p is even, so that (2) equals (1) and our result is optimal in this case. If p is odd, then $\epsilon = 1$, so that the only missing case is the one where equality is attained in (1). The pairs (p,k,δ) for which this happens are completely described in Proposition 7.5. These cases are more tricky and left to future research.

The proof of Theorem 0.1 relies on a degeneration argument, which requires a rather delicate treatment. We use a well known degeneration of S, embedded in $\mathbb{P}^p$ via $|H|$, to a general union S_0 of two smooth rational normal scrolls intersecting transversally along an elliptic curve of degree $p+1$ (see §1.5). In §§4 and 5 we describe nodal curves on S_0 that fill up limit components of $V_{|H|,\delta}$ and in §6 we describe possible limits of $V^k_{|H|,\delta}$. If nonempty, these limit varieties have the expected dimension, and this yields nonemptiness and expected dimension for the general S (see Proposition 1.5). In §7 we show nonemptiness of the limit varieties and the required properties for the g^1_k's.

Besides its intrinsic interest for Brill-Noether theory and moduli problems, the subject of this paper is related to the study of Mori theory and rational curves on the $2k$-dimensional hyperkähler manifold $\mathrm{Hilb}^k(S)$ parametrizing length k-subschemes of the $K3$ surface S. A curve with k-gonal normalization on S determines a rational curve on $\mathrm{Hilb}^k(S)$. For the importance of rational curves on hyperkähler manifolds see, e.g., [Hu1, Hu2, Bo, HT3, HT2, HT1, W2, W1, WW] and § 8. In particular, rational curves determine the nef and ample cones, and thus generate the Mori cone of such a manifold.

If X is a hyperkähler manifold, $H^2(X,\mathbb{Z})$ is endowed with the so-called *Beauville-Bogomolov quadratic form* which induces a quadratic form on $\mathrm{Pic}(X)$ and, by duality, on $H_2(X,\mathbb{Z})$ and therefore on the vector space $N_1(X)$ of 1-cycles. We will call the value of this form on a cycle class the *Beauville-Bogomolov self-intersection* of that cycle (see §§2, 8). In §2 we determine the classes in $N_1(\mathrm{Hilb}^k(S))$ of the rational curves in $\mathrm{Hilb}^k(S)$ corresponding to the curves on S in Theorem 0.1 (see Lemma

2.1) and also compute the corresponding Beauville-Bogomolov self-intersections (see (29)). Here the properties stated at the end of the theorem play an essential role. The lower δ is, the lower the self-intersection is, and the closer the class is to the boundary of the Mori cone. As a consequence, we obtain necessary conditions for a divisor in $\mathrm{Hilb}^k(S)$ to be nef or ample (see Proposition 8.2). In certain cases we also prove that the classes of the rational curves in $\mathrm{Hilb}^k(S)$ we obtain via Theorem 0.1 generate the boundary of the Mori cone of $\mathrm{Hilb}^k(S)$ (see Corollary 8.4 and Proposition 8.8).

In §8 we show that the rational curves in $\mathrm{Hilb}^k(S)$ corresponding to the curves in Theorem 0.1 with the lowest values of δ fit in an interesting conjecture by Hassett and Tschinkel, in the sense that they conjecture curves with such negative Beauville-Bogomolov self-intersections to exist and their classes to generate extremal rays. Further discussions and refinements of their conjectures are also contained in § 8.

Nontrivial divisors with Beauville-Bogomolov self-intersection zero are the objects of conjectures by Huybrechts and Sawon to the effect that their existence should imply that the hyperkähler manifold is birational to a Lagrangian fibration and that it *is* a Lagrangian fibration if the divisors are nef (see again §8). Sawon [Sa2] has proved that $\mathrm{Hilb}^k(S)$, where S is $K3$, is a Lagrangian fibration in many cases. With the help of the rational curves obtained from Theorem 0.1, we prove that isotropic divisors are not nef under certain numerical conditions, thus obtaining additional *necessary conditions* (on p and k) for $\mathrm{Hilb}^k(S)$ to be a Lagrangian fibration (see Corollary 8.11). As a conjecture, we propose that these conditions are sufficient.

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1. Severi varieties, $K3$ surfaces and k-gonal loci

1.1. **Severi varieties and k-gonal loci.** Let S be a connected, projective surface with normal crossing singularities and let $|H|$ be a base point free, complete linear system of Cartier divisors on S whose general element is a connected curve H with at most nodes as singularities, located at the singular points of S. We will set $p = p_a(H)$.

For any integer $0 \leq \delta \leq p$, we denote by $V_{|H|,\delta}(S)$, or simply by $V_{|H|,\delta}$, the locally closed subscheme of $|H|$ parametrizing the universal family of curves $C \in |H|$ having only nodes as singularities, δ of them (called the *marked nodes*) off the singular locus of S, and such that the partial normalization $\widetilde{C}$ at these δ nodes is connected (i.e, the marked nodes are *not disconnecting nodes*). We set $g = p - \delta = p_a(\widetilde{C})$. If S is smooth the $V_{|H|,\delta}$'s are called *Severi varieties* of δ-nodal curves in $|H|$ on S. We use the same terminology in our more general setting.

Let $g \geq 3$ be an integer. We denote by $\mathcal{M}_g$ the *moduli space (or stack)* of smooth curves of genus g, whose dimension is $3g-3$. We recall that $\mathcal{M}_g$ is quasi-projective and admits a projective compactification $\overline{\mathcal{M}_g}$, parametrizing all connected stable curves of arithmetic genus g.

One has the *moduli morphism*

$$\psi_{S,H,\delta} : V_{|H|,\delta}(S) \longrightarrow \mathcal{M}_g \tag{3}$$

sending $C \in V_{|H|,\delta}$ to the isomorphism class of the stable model $\overline{C}$ of the partial normalization $\widetilde{C}$ of C at the δ marked nodes. We write ψ rather than $\psi_{S,H,\delta}$ if no confusion arises. If ψ is generically finite to its image, we say that $V_{|H|,\delta}(S)$ has *maximal number of moduli g*.

One can consider the stratification of $\mathcal{M}_g$ in terms of gonality

$$\mathcal{M}^1_{g,2} \subset \mathcal{M}^1_{g,3} \subset \ldots \subset \mathcal{M}^1_{g,k} \subset \ldots \subset \mathcal{M}_g,$$

where

$$\mathcal{M}^1_{g,k} := \Big\{[C] \in \mathcal{M}_g |\; C \text{ possesses a } g^1_k\Big\},$$

called the *k-gonal locus* in $\mathcal{M}_g$, is irreducible, of dimension $2g+2k-5$ when $g \geq 2(k-1)$, whereas $\mathcal{M}^1_{g,k} = \mathcal{M}_g$ when $g \leq 2(k-1)$ (see e.g. [AC]). For any integer $k \geq 2$, we define

$$V^k_{|H|,\delta} := \left\{ C \in V_{|H|,\delta} \mid \psi(C) \in \overline{\mathcal{M}^1_{g,k}} \right\}$$

which has a natural scheme structure. This is called the *k-gonal locus* inside $V_{|H|,\delta}$. Recall that $\psi(C) \in \overline{\mathcal{M}^1_{g,k}}$ if and only if the partial normalization $\widetilde{C}$ of C at the δ marked nodes is stably equivalent to a curve that is the domain of an admissible cover of degree k to a stable pointed curve of genus 0 (see [HM, Theorem (3.160)]).

1.2. $K3$ **surfaces.** We will mainly consider the case in which S is a smooth, projective $K3$ surface, endowed with a globally generated *primitive*, i.e. indivisible, divisor H with $p = p_a(H) \geq 2$. We call (S,H) a *primitive (or primitivley polarized)* $K3$ *surface of genus* p. We denote by $\mathcal{K}_p$ the *moduli space (or stack)* of primitive $K3$ surfaces of genus p, which is smooth, of dimension 19, and the general element (S,H) is such that $\mathrm{Pic}(S)$ is generated by the class of $\mathcal{O}_S(H)$ and H is very ample.

If $V_{|H|,\delta}(S) \neq \emptyset$, then it is *regular*, i.e. it is smooth and of the *expected dimension* g. Indeed, the marked, not disconnecting nodes of the curves in $V_{|H|,\delta}$ impose independent conditions to the linear system $|H|$ (see e.g. [C-S]). If $V_{|H|,\delta} \neq \emptyset$ and $\delta' < \delta$, then $V_{|H|,\delta} \subset \overline{V_{|H|,\delta'}}$.

Remark 1.1. The latter holds for $V_{|H|,\delta}(S)$ also when S is a connected surface with local normal crossing singularities, trivial dualizing bundle, $h^1(S,\mathcal{O}_S) = 0$ and H a globally generated, primitive divisor on S. Indeed, the usual arguments (like in [C-S]) apply with no change.

By a result of Mumford's (cf. [MM, Appendix]), for all $\delta \leq p$ the Severi varieties $V_{|H|,\delta}$ are nonempty. Chen extended this to Severi varieties $V_{|mH|,\delta}$ with $m > 1$ (cf. [Ch]).

Proposition 1.2. *Let $(S,H) \in \mathcal{K}_p$. The differential of the moduli morphism $\psi_{S,H,\delta}$ is everywhere injective, hence all components of $V_{|H|,\delta}(S)$ have maximal number of moduli.*

Proof. Let C be a curve in $V_{|H|,\delta}(S)$, let $f : \widetilde{C} \to C$ be the normalization at the δ nodes. We have the following exact sequence

$$0 \longrightarrow T_{\widetilde{C}} \longrightarrow f^*(T_S) \longrightarrow N_f \longrightarrow 0,$$

which defines the normal sheaf N_f to the map f. The differential of ψ at $\widetilde{C}$ is the coboundary map $H^0(\widetilde{C}, N_f) \to H^1(\widetilde{C}, T_{\widetilde{C}})$. Hence it suffices to prove that $h^0(\widetilde{C}, f^*(T_S)) = 0$, i.e. that $h^0(\widetilde{C}, \varphi^*(T_S)_{|\widetilde{C}}) = 0$, where $\varphi : \widetilde{S} \to S$ is the blow-up of S at the nodes of C. Denote by E_i the exceptional divisors, with $1 \leq i \leq \delta$. Consider the diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{\widetilde{S}}(-\widetilde{C}) \arrow[r] \arrow[d] & \varphi^*(T_S)(-\widetilde{C}) \arrow[r] \arrow[d] & \oplus_{i=1}^{\delta} \mathcal{O}_{\mathbb{P}^1}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{\widetilde{S}} \arrow[r] & \varphi^*(T_S) \arrow[r] \arrow[d] & \oplus_{i=1}^{\delta} \mathcal{O}_{\mathbb{P}^1}(1) \arrow[r] & 0 \\
 & & \varphi^*(T_S)_{|\widetilde{C}} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

One has $h^0(\widetilde{S}, \varphi^*(T_S)) = h^0(S, T_S) = 0$ and $H^1(\widetilde{S}, \varphi^*(T_S)(-\widetilde{C})) \simeq H^1(\widetilde{S}, T_{\widetilde{S}}(-\widetilde{C}))$. The map $H^1(\widetilde{S}, T_{\widetilde{S}}(-\widetilde{C})) \to H^1(\widetilde{S}, T_{\widetilde{S}})$ is injective, because of its obvious interpretation in terms of deformations, and its image $\mathfrak{A}$ correspond to first order deformations of $\widetilde{S}$ that keep $\widetilde{C}$ fixed. These deformations do not move the E_i's, and therefore $\mathfrak{A}$ intersects the image of $H^0(\widetilde{S}, \oplus_{i=1}^{\delta} \mathcal{O}_{\mathbb{P}^1}(1)) \to$

$H^1(\widetilde{S}, T_{\widetilde{S}})$ in (0). This implies that the map $H^1(\widetilde{S}, \varphi^*(T_S)(-\widetilde{C})) \to H^1(\widetilde{S}, \varphi^*(T_S))$ is injective and $h^0(\widetilde{C}, \varphi^*(T_S)_{|\widetilde{C}}) = 0$ follows. □

1.3. **Universal Severi varieties and degenerations.** For any $p \geq 2$, and $0 \leq \delta \leq p$, one can consider a stack $\mathcal{V}_{p,\delta}$ (see [FKP3, Proposition 4.8]), called the *universal Severi variety*, which is pure and smooth of dimension $19 + g$, endowed with a morphism $\phi_{p,\delta} : \mathcal{V}_{p,\delta} \to \mathcal{K}_p^\circ$, where $\mathcal{K}_p^\circ$ is a suitable dense open substack of $\mathcal{K}_p$ and its fibres are so described

$$\begin{array}{ccc} \mathcal{V}_{p,\delta} & \supset & V_{|H|,\delta}(S) \\ {\scriptstyle \phi_{p,\delta}}\downarrow & & \downarrow \\ \mathcal{K}_p^\circ & \ni & (S,H) \end{array}$$

The morphism $\phi_{p,\delta}$ is smooth on all components of $\mathcal{V}_{p,\delta}$, each dominating $\mathcal{K}_p^\circ$.

In a similar way one can consider the *k-gonal universal locus* $\mathcal{V}^k_{p,\delta} \subseteq \mathcal{V}_{p,\delta}$.

We will need this in a more general setting. Suppose we have a proper flat family of surfaces $f : \mathcal{S} \to \mathbb{D}$, where $\mathbb{D}$ is a disc. Assume that:

- $\mathcal{S}$ is smooth, endowed with a line bundle $\mathcal{H}$;
- f is smooth over $\mathbb{D}^* = \mathbb{D} - \{0\}$;
- if $t \in \mathbb{D}^*$, then the fibre S_t of f over t is a $K3$ surface;
- the fibre S_0 of f over 0 is a local normal crossing divisor in $\mathcal{S}$;
- the line bundle $\mathcal{H}_t := \mathcal{H}_{|S_t}$ determines a complete linear system $|H_t|$ of dimension p for all $t \in \mathbb{D}$ and $(S_t, H_t) \in \mathcal{K}_p$ for all $t \in \mathbb{D}^*$.

Since $\mathcal{V}_{p,\delta}$ is functorially defined, we have *f-relative Severi varieties* $\phi_{f;p,\delta} : \mathcal{V}_{f;p,\delta} \to \mathbb{D}^*$, with $\mathcal{V}_{f;p,\delta}$ locally closed in $\mathbb{P}(f_*(\mathcal{H}))$, such that the fibre of $\phi_{f;p,\delta}$ over t is $V_{|H_t|,\delta}(S_t)$ for all $t \in \mathbb{D}^*$. We will drop the index δ when $\delta = 0$.

Lemma 1.3. *Let $C_0 \in |H_0|$ be an element of $V_{|H_0|,\delta}(S_0)$, with δ not disconnecting nodes $q_1, \ldots, q_\delta$ off the singular locus of S_0. Then C_0 sits in the closure of $\mathcal{V}_{f;p,\delta}$ in $\mathbb{P}(f_*(\mathcal{H}))$.*

Proof. We have a commutative diagram with exact rows and column

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
0 \arrow[r] & T_{[C_0]}V_{|H_0|,\delta}(S_0) \arrow[r] & \mathbb{C}^p \simeq T_{[C_0]}|H_0| \simeq H^0(C_0, N_{C_0/S_0}) \arrow[r, "\alpha"] \arrow[d] & \oplus_{i=1}^{\delta} T^1_{q_i} \arrow[d, equal] \\
0 \arrow[r] & T_{[C_0]}\mathcal{V}_{f;p,\delta} \arrow[r] & \mathbb{C}^{p+1} \simeq T_{[C_0]}\mathcal{V}_{f;p} \simeq H^0(C_0, N_{C_0/\mathcal{S}}) \arrow[r, "\beta"] \arrow[d] & \oplus_{i=1}^{\delta} T^1_{q_i} \\
 & & H^0(C_0, \mathcal{O}_{C_0}) \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

By hypothesis, α is onto, hence so is β. Thus $\dim \mathcal{V}_{f;p,\delta} = \dim V_{|H_0|,\delta}(S_0) + 1$, and the assertion follows. □

1.4. $K3$ **surfaces and k-gonal loci.** Let $(S,H) \in \mathcal{K}_p$ be general. By Brill-Noether theory, $V^k_{|H|,\delta} = V_{|H|,\delta}$ if $\delta \geq p - 2(k-1)$.

Proposition 1.4. *Let (S,H) be in $\mathcal{K}_p$. Assume $g := p - \delta > 2(k-1)$. Then for any irreducible component V of $V^k_{|H|,\delta}$ one has $\dim(V) \geq 2(k-1)$.*

Proof. Consider the morphism ψ in (3). Let V be an irreducible component of $V^k_{|H|,\delta}$ and V' the g-dimensional, irreducible component of $V_{|H|,\delta}$ containing it, so that

$$\emptyset \neq \psi(V) \subseteq \psi(V') \cap \mathcal{M}^1_{g,k}.$$

Let W be any irreducible component of $\psi(V') \cap \overline{\mathcal{M}^1_{g,k}}$ such that $\psi(V) \subseteq W$. One has

$$\begin{aligned} \dim(W) &\geq \dim(\psi(V')) + \dim(\mathcal{M}^1_{g,k}) - \dim(\mathcal{M}_g) \\ &= \dim(\psi(V')) + 2(k-1) - g. \end{aligned}$$

Since the dimension of the general fiber of $\psi_{|V'}$ over $\psi(V')$ is $g - \dim(\psi(V'))$, we have

$$\dim(V) \geq \dim(W) + (g - \dim(\psi(V'))) = 2(k-1).$$

□

The proof of Proposition 1.4 shows that the *expected dimension* of an irreducible component of $V^k_{|H|,\delta}$ is $\min\{2(k-1), p-\delta\}$.

It is convenient to have a *relative version* of Proposition 1.4. Let $f : \mathcal{S} \to \mathbb{D}$ be as in §1.3. One can define the *f-relative k-gonal locus* $\mathcal{V}^k_{f;p,\delta} \subseteq \mathcal{V}_{f;p,\delta}$ over $\mathbb{D}^*$.

Proposition 1.5. *Let V_0 be a component of $V^k_{|H_0|,\delta}(S_0)$ contained in the closure of $\mathcal{V}_{f;p,\delta}$. If $\dim(V_0) = 2(k-1)$, then V_0 is contained in an irreducible component $\mathcal{V}$ of $\mathcal{V}^k_{f;p,\delta}$ dominating $\mathbb{D}^*$, with* $\dim(\mathcal{V}) = \dim(V_0) + 1$.

Proof. Similar to the proof of Proposition 1.4 (see also [FKP1, Prop. 5.11 and its proof]). □

1.5. **Unions of scrolls as limits of $K3$ surfaces.** If $(S,H) \in \mathcal{K}_p$, then $|H|$ determines a morphism $\phi_{|H|} : S \to \mathbb{P}^p$, which is an embedding for general (S,H). Let $\mathfrak{H}_p$ be the component of the Hilbert scheme of surfaces in $\mathbb{P}^p$ whose general point corresponds to an embedding of an S as above. One has $\dim(\mathfrak{H}_p) = p^2 + 2p + 19$ and $\mathfrak{H}_p$ is smooth at each point corresponding to a smooth $K3$ surface. The component $\mathfrak{H}_p$ contains points which correspond to *degenerations* of elements of $\mathcal{K}_p$, as we will now explain (see [CLM1, §2.2]).

Fix $p \geq 4$ and set $n := \lfloor p/2 \rfloor$. Let $E \subset \mathbb{P}^p$ be a smooth elliptic normal curve of degree $p+1$. Fix two general line bundles $L_i \in \mathrm{Pic}^2(E)$, with $i = 1, 2$. Let R_1 and R_2 be the rational normal scrolls of degree $p-1$ in $\mathbb{P}^p$ *defined by* L_1 and L_2, respectively, i.e. R_i is the union of lines spanned by the divisors of $|L_i|$. Note that $R_1 \simeq R_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ if p is odd, whereas $R_1 \simeq R_2 \simeq \mathbb{F}_1$ if p is even. We let $\mathfrak{s}_i$ and $\mathfrak{f}_i$ denote the classes of the nonpositive section and fiber, respectively, of R_i. By the generality of L_1, L_2, one has $R_1 \cap R_2 = E$, the intersection is transversal and E is anticanonical on each R_i. Moreover $R = R_1 \cup R_2$ corresponds to a smooth point of $\mathfrak{H}_p$ and $\mathcal{O}_{R_i}(1) \sim \mathfrak{s}_i + n\mathfrak{f}_i$.

Let $\mathbb{D}$ be a disc and $\varphi : \mathbb{D} \to \mathfrak{H}_p$ a holomorphic map with nonzero differential, such that $\varphi(0)$ is the point corresponding to R and, for $t \in \mathbb{D}$ general, $\varphi(t)$ is a general element in $\mathfrak{H}_p$. By pulling back the universal family on $\mathfrak{H}_p$, we obtain a flat family $\mathcal{X} \to \mathbb{D}$, whose total space has 16 ordinary double points along E in the central fibre, and is otherwise smooth. The singular points are the zero locus of the section in $H^0(E, T^1_R)$ corresponding to the section in $H^0(R, N_{R|\mathbb{P}^g})$ that is the image of the differential of φ at 0 (cf. [CLM1, p. 647] or [Ch, Sec. 3.1]). Note that T^1_R is a line bundle of degree 16 on E.

Remark 1.6. Let $\mathfrak{R}_p$ be the locus of pairs of scrolls inside $\mathfrak{H}_p$. One has $\mathrm{codim}_{\mathfrak{H}_p}(\mathfrak{R}_p) = 16$. Since the general point R of $\mathfrak{R}_p$ is smooth for $\mathfrak{H}_p$, we can look at the 16-dimensional *normal space* to $\mathfrak{R}_p$ at R, which coincides with $H^0(E, T^1_R)$. Hence, if ϕ is general, the 16 singular points of $\mathcal{X}$ form a general divisor $Z \in |T^1_R|$.

One can perform a small resolution of the singularities of $\mathcal{X}$, obtaining a new family $f : \mathcal{S} \to \mathbb{D}$, which has all properties indicated in §1.3, including the existence of the *hyperplane bundle* $\mathcal{H}$. The central fibre S_0 however is no longer R, but a modification of it. Precisely one can work things out in such a way that $S_0 = R_1 \cup \widetilde{R}_2$, where $\widetilde{R}_2$ is the blow up of R_2 at Z, and R_1 and $\widetilde{R}_2$ are glued along $E \subset R_1$ and its strict transform (still denoted by E) on $\widetilde{R}_2$.

2. Rational curves in the Hilbert scheme of points of a $K3$ surface

If S is a $K3$ surface, then $\mathrm{Hilb}^k(S)$ is a *hyperkähler manifold*, also called an *irreducible symplectic manifold* (see e.g. [Be, Hu1]). The cohomology group $H^2(\mathrm{Hilb}^k(S),\mathbb{Z})$ is endowed with the *Beauville-Bogomolov quadratic form* q and one has the orthogonal decomposition

$$H^2(\mathrm{Hilb}^k(S),\mathbb{Z}) \simeq H^2(S,\mathbb{Z}) \oplus_\perp \mathbb{Z}[\mathfrak{e}_k], \tag{4}$$

where $\Delta_k := 2\mathfrak{e}_k$ is the class of the divisor parametrizing nonreduced 0-dimensional subschemes [Be], and Δ_k is also the exceptional locus of the *Hilbert-Chow morphism* $\mu_k : \mathrm{Hilb}^k(S) \to \mathrm{Sym}^k(S)$. The embedding of $H^2(S,\mathbb{Z})$ into $H^2(\mathrm{Hilb}^k(S),\mathbb{Z})$ in (4) is given by sending a homology class $F \in H^2(S,\mathbb{Z})$ to the class in $H^2(\mathrm{Hilb}^k(S),\mathbb{Z})$ parametrizing subschemes whose support intersects F. By abuse of notation we will still denote by F this class in $H^2(\mathrm{Hilb}^k(S),\mathbb{Z})$. The restriction of the Beauville-Bogomolov form to $H^2(S,\mathbb{Z})$ is the cup product on S, and $q(\mathfrak{e}_k) = -2(k-1)$. Accordingly, (4) induces an orthogonal decomposition (see [Be]):

$$\mathrm{Pic}(\mathrm{Hilb}^k(S)) \simeq \mathrm{Pic}(S) \oplus_\perp \mathbb{Z}[\mathfrak{e}_k]. \tag{5}$$

Given a primitive class $\alpha \in H_2(\mathrm{Hilb}^k(S),\mathbb{Z})$, there exists a unique class $w_\alpha \in H^2(\mathrm{Hilb}^k(S),\mathbb{Q})$ such that $\alpha \cdot v = q(w_\alpha, v)$, for all $v \in H^2(\mathrm{Hilb}^k(S),\mathbb{Z})$, and one sets (cf. e.g. [HT1])

$$q(\alpha) := q(w_\alpha). \tag{6}$$

This gives a $\mathbb{Q}$-valued form on homology, and we have

$$H_2(\mathrm{Hilb}^k(S),\mathbb{Z}) \simeq H_2(S,\mathbb{Z}) \oplus_\perp \mathbb{Z}[\mathfrak{r}_k], \tag{7}$$

where $\mathfrak{r}_k$ is characterized as the homology class orthogonal to $H^2(S,\mathbb{Z})$ and satisfying $\mathfrak{e}_k \cdot \mathfrak{r}_k = -1$, see e.g. [HT3, §1]. As explained in [HT1, Ex. 4.2], $\mathfrak{r}_k$ is the class of a fiber of the Hilbert-Chow morphism, i.e., it is the class of the rational curve in Δ_k corresponding to the curve lying above $2x_1 + x_2 + \cdots + x_{k-1}$ in $\mathrm{Sym}^k(S)$, for any $k-1$ distinct points $x_1, \ldots, x_{k-1}$ of S. The embedding of $H_2(S,\mathbb{Z})$ in $H_2(\mathrm{Hilb}^k(S),\mathbb{Z})$ is given by sending the class of a cycle Y to the class of the cycle

$$\Big\{\xi \in \mathrm{Hilb}^k(S) \,|\, \mathrm{Supp}(\xi) = \{p_1, \ldots, p_{k-1}, y\},\ y \in Y\Big\},$$

where $p_1, \ldots, p_{k-1}$ are distinct fixed points of S off Y.

The decomposition (7) induces

$$N_1(\mathrm{Hilb}^k(S),\mathbb{Z}) \simeq \mathrm{Pic}(S) \oplus_\perp \mathbb{Z}[\mathfrak{r}_k].$$

If $R \equiv D - y\mathfrak{r}_k$ in $N_1(\mathrm{Hilb}^k(S),\mathbb{Z})$, with $D \in \mathrm{Pic}(S)$, then

$$w_R = D - \frac{y}{2(k-1)}\mathfrak{e}_k,$$

and by (6), one has

$$q(R) = D^2 - \frac{y^2}{2(k-1)}. \tag{8}$$

We mentioned already in the Introduction the importance of rational curves on hyperkähler manifolds. The relation with the topic of this paper is that a curve C on a $K3$ surface whose normalization $\widetilde{C}$ possesses a g^1_k gives rise, in an obvious way, to an irreducible rational curve R in $\mathrm{Hilb}^k(S)$. Indeed, the $g^1_k = |A|$ on $\widetilde{C}$ induces a $\mathbb{P}^1_{(C,A)} \subset \mathrm{Sym}^k(\widetilde{C})$ and this is mapped to an irreducible rational curve $\overline{R}_{(C,A)} \subset \mathrm{Sym}^k(S)$ by the composed morphism

$$\mathrm{Sym}^k(\widetilde{C}) \longrightarrow \mathrm{Sym}^k(C) \hookrightarrow \mathrm{Sym}^k(S).$$

The irreducible rational curve $R = R_{(C,A)} \subset \mathrm{Hilb}^k(S)$ is the strict transform $(\mu_k)^{-1}_* \overline{R}_{(C,A)}$ by the Hilbert-Chow morphism.

Let C be an element of $V^k_{|H|,\delta}$, and assume that its normalization carries a g^1_k satisfying:

(9) all the nodes of C are non-neutral with respect to the g^1_k;

(10) the g^1_k has only simple ramification.

Let R be the corresponding rational curve in $\mathrm{Hilb}^k(S)$.

Lemma 2.1. *Under hypotheses* (9) *and* (10), *the class*[1] *of R in $N_1(\mathrm{Hilb}^k(S),\mathbb{Z})$ is $H-(g+k-1)\mathfrak{r}_k$.*

Proof. Write $R = H - y\mathfrak{r}_k$, so that $y = \mathfrak{e}_k \cdot R$. Since all nodes are non-neutral and the g^1_k has simple ramification everywhere, by Riemann-Hurwitz we have

$$y = \mathfrak{e}_k \cdot R = \frac{1}{2}\Delta_k \cdot R = \frac{1}{2}(2g + 2k - 2) = p - \delta + k - 1.$$

□

The particular case $p = 9$, $\delta = 2$ and $k = 4$ is treated in [HT1, Ex. 4.5].

Remark 2.2. Independently of the singularities of S and of the ramification, if R comes from a g^1_k on $\widetilde{C}$, the same proof yields the class $R = H - y\mathfrak{r}_k$, where $y \le g + k - 1$. In addition, if C is nodal, and δ_0 is the number of non-neutral nodes, one has $y \le p - \delta_0 + k - 1$, with equality in the case of simple ramification.

3. Necessary conditions for existence of linear series on normalizations

Consider the usual *Brill-Noether number* $\rho(g, r, d) = g - (r + 1)(r + g - d)$.

Theorem 3.1. *Let S be a $K3$ surface with $\mathrm{Pic}\, S \simeq \mathbb{Z}[H]$ and let $p = p_a(H)$. Assume that $C \in |H|$ is a curve whose normalization possesses a g^r_d. Let g be the geometric genus of C and $\delta = p - g$ and set $\alpha := \left\lfloor \frac{gr+(d-r)(r-1)}{2r(d-r)} \right\rfloor$. Then*

$$\rho(p, \alpha r, \alpha d + \delta) \ge 0, \ \textit{i.e.}\ \ \delta \ge \alpha\Big(rg - (d - r)(\alpha r + 1)\Big). \tag{11}$$

Proof. Let $\nu : \widetilde{C} \to C$ be the normalization of C and let A be the line bundle such that $|A| = g^r_d$. Then $|lA|$ contains a g^{lr}_{ld} for any positive integer l, and one has $\overline{W}^{lr}_{ld+\delta}(C) \neq 0$, where $\overline{W}^s_n(C)$ is the *generalized Brill-Noether locus* parametrizing torsion free sheaves of degree n and at least $s+1$ sections (see, e.g., [FKP2, Prop. 3.2]).

We claim that

$$\rho(p, lr, ld + \delta) = l^2 r(d - r) - l(gr + r - d) + \delta \ge 0. \tag{12}$$

Indeed, any $\mathcal{A} \in \overline{W}^{lr}_{ld+\delta}(C)$ gives rise to a rank $lr + 1$ vector bundle $\mathcal{E}$ on S with $c_1(\mathcal{E}) = C$, $c_2(\mathcal{E}) = lr + \delta$ (see [Go]). One has $\chi(\mathcal{E} \otimes \mathcal{E}^*) = 2(1 - \rho(p, lr, ld + \delta))$ (see e.g. [La, § 1]). As in the proof of [La, Lemma 1.3], if $\mathcal{E}$ had nontrivial endomorphisms, the linear system $|C|$ would contain a reducible curve. Therefore $h^0(\mathcal{E} \otimes \mathcal{E}^*) = h^2(\mathcal{E} \otimes \mathcal{E}^*) = 1$, so that $\chi(\mathcal{E} \otimes \mathcal{E}^*) \le 2$, implying (12).

The quadratic polynomial in l in (12) attains its minimum for $l_0 = \frac{gr+r-d}{2r(d-r)}$. The inequality (12) must hold for the closest integer to l_0, which is α. This proves (11). □

Remark 3.2. Fix p, d and r. If (11) is satisfied by $\delta = \delta_0$, then it is also satisfied for all $\delta \ge \delta_0$. Indeed, the inequality (11) is equivalent to $\rho(p, lr, ld + \delta) \ge 0$ for all positive integers l and for any $\delta' \ge \delta$, we have $\rho(p, lr, ld + \delta') \ge \rho(p, lr, ld + \delta)$.

Next we concentrate on the case $r = 1$ and set $d = k$, where (11) reads like (1).

Remark 3.3. Set $\rho = \rho(p, \alpha, k\alpha + \delta)$. It is convenient to write (1) in the form

$$\rho \ge 0 \ \text{ i.e. } \ \delta \ge \frac{(g - k + 1)^2 - \beta^2}{4(k - 1)}, \tag{13}$$

where $\beta := (k - 1)(2\alpha + 1) - g$, whence $-(k - 1) < \beta \le k - 1$.

We obtain a bound on the *Beauville-Bogomolov self-intersection* of the rational curves in $\mathrm{Hilb}^k(S)$ corresponding to the curves in Theorem 3.1, which confirms for them a conjecture by Hassett and Tschinkel [HT1, Conj. 1.2].

[1]There is an erroneous fraction of 1/2 in the corresponding formula for $k = 2$ in [FKP3, (6.7)], due to a trivial computational mistake in the line above: $\mathbb{P}^1_\Delta.\mathfrak{e} = -2$ should have been -1, where $\mathbb{P}^1_\Delta$ is $\mathfrak{r}_k$ in our notation.

Corollary 3.4. *Let S be a $K3$ surface with $\operatorname{Pic} S \simeq \mathbb{Z}[H]$ and let $p = p_a(H)$. Assume that $C \in |H|$ is a curve whose normalization has genus g and possesses a $g^1_k = |A|$. Let $R = R_{(C,A)}$. Then*

$$q(R) \geq 2(p-1) - \frac{(g+k-1)^2}{2(k-1)} = 2(\rho - 1) - \frac{\beta^2}{2(k-1)} \geq -\frac{k+3}{2}, \tag{14}$$

with ρ and β as in Remark 3.3. Equality holds on the left under hypotheses (9) *and* (10).

Proof. By Remark 2.2, the class of R is $H - y\mathfrak{r}_k$ where $y \leq g+k-1$ and by (8) one has $q(R) = 2(p-1) - \frac{y^2}{2(k-1)}$. Then (14) follows by Theorem 3.1 and Remark 3.3. (The middle equality is a direct computation.) The last assertion follows from Lemma 2.1. □

4. Special chains of rational curves

In this section we introduce the building blocks of limits on R of nodal curves on the general surface in $\mathfrak{H}_p$ in the degeneration described in § 1.5.

4.1. **Odd chains.** Let m be a positive integer satisfying $m \leq n$. An *odd chain of length* $2m-1$ *based on* R_1 is a sum of $2m-1$ distinct lines

$$f_{1,1} + f_{2,1} + f_{1,2} + f_{2,2} + \cdots + f_{1,m-1} + f_{2,m-1} + f_{1,m}, \; f_{i,j} \in |\mathfrak{f}_i|,$$

where $f_{2,j}$ intersects only $f_{1,j}$ and $f_{1,j+1}$, for $j = 1, \ldots, m-1$. The chain intersects E in $2m$ points, consisting of m divisors of $|L_1|$, and $2m-2$ of them lie on the intersections between two lines, whereas the remaining two are on $f_{1,1}$ and $f_{1,m}$. This pair of points will be called *the distiguished pair of points of the chain.* Similarly we define odd chains of length $2m-1$ based on R_2 with their distiguished pairs of points.

We will denote by $\mathcal{C}_{i,m}$ the family of odd chains of length $2m-1$ based on R_i. Note that $\mathcal{C}_{i,m}$ is a locally closed subvariety of a Hilbert scheme of curves on R.

Lemma 4.1. *The map sending an odd chain of length $2m-1$ based on R_i, $i = 1, 2$, to its pair of distinguished points on E is a birational morphism between $\mathcal{C}_{i,m}$ and $|mL_i - (m-1)L_{3-i}|$.*

Proof. We describe the inverse map. Its existence is obvious if $m = 1$. We proceed by induction on m. Let $a + b \in |mL_i - (m-1)L_{3-i}|$ be a general point. Then $|L_i - a| = \{a'\}$ and $|L_i - b| = \{b'\}$. Therefore $a' + b' \sim (m-1)L_{3-i} - (m-2)L_i$ and we are done by induction. □

Assume that the chain of length $2m-1$ based on R_i is contained in a nodal hyperplane section H of R. Let Γ_1 and Γ_2 be the sections on R_1 and R_2 contained in H. Then the distiguished points are $a_{3-i} := \Gamma_{3-i} \cap f_{i,1}$ and $b_{3-i} := \Gamma_{3-i} \cap f_{i,m}$. We will call the 2-cycle $a_{3-i} + b_{3-i}$ *the 2-cycle on* Γ_{3-i} *associated to the chain.* The chain intersects Γ_1 and Γ_2 in a total of $2m-1$ points. They will be called *the nodes of H associated to the chain*, and are distributed as m nodes on the surface R_i on which the chain is based, and $m-1$ on the other.

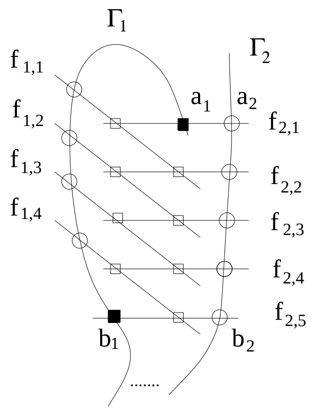

FIGURE 1.

Figure 1 shows an odd chain of length 9 based on R_2 contained in a hyperplane section. All intersection points between the chain and E are marked with a box, the two distinguished points are marked with filled boxes, the nodes associated to the chain are marked with circles.

In Figure 2 we describe the stable model of the partial normalization at the associated nodes of a hyperplane section containing an odd chain: in the stable model all lines are contracted, so that the two distiguished points of the chain lying on the same Γ_i are identified, creating a node.

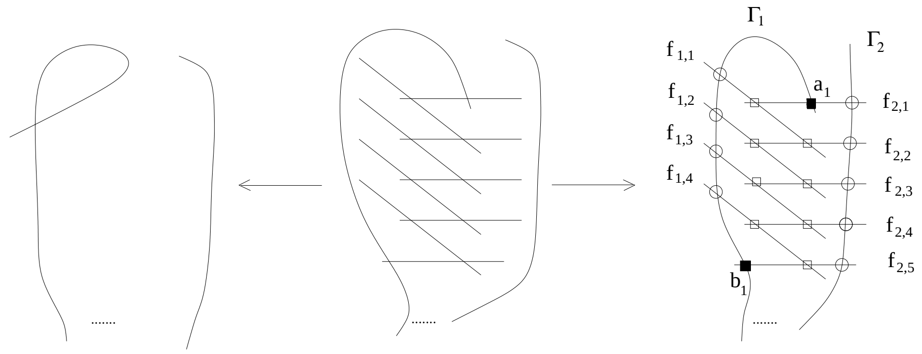

FIGURE 2.

4.2. **Even chains.** An *even chain of length* $2m$ is a sum of $2m$ distinct lines

$$f_{1,1}+f_{2,1}+f_{1,2}+f_{2,2}+\cdots+f_{1,m}+f_{2,m},\ f_{i,j}\in|\mathfrak{f}_i|,$$

where each $f_{1,j}$ intersects $f_{2,j}$ and $f_{2,j+1}$, $j=1,\ldots,m-1$. The chain intersects E in $2m+1$ points, $2m-1$ of which lie on the intersections between two lines. Upon renumbering, we choose the convention that the remaining two points are on $f_{2,1}$ and $f_{1,m}$, and we denote them by $a_{1,1}$ and $a_{2,1}$ (note the order), respectively. They will be called the two *distiguished points of the chain* and the lines $f_{2,1}$ and $f_{1,m}$ will be called the *distiguished lines of the chain.*

We will denote by c_i, $i=1,2$, the unique point such that $a_{i,1}+c_i\in|L_{3-i}|$. The $2m+1$ intersection points of the chain with E form a divisor in $|mL_i+a_{i,1}|$, $i=1,2$.

We will denote by $\mathcal{C}_m$ the family of even chains of length $2m$.

Lemma 4.2. *(a) The map sending an even chain of length $2m$ to the pair $(a_{1,1},a_{2,1})$ in $E\times E$, is a birational morphism of $\mathcal{C}_m$ to the closure of its image, which is the graph G_m of the translation by $m(L_1-L_2)$.*

(b) The map sending an even chain of length $2m$ to

$$(a_{1,1}+c_2,a_{2,1}+c_1)\in|mL_2-(m-1)L_1|\times|mL_1-(m-1)L_2|\simeq\mathbb{P}^1\times\mathbb{P}^1,$$

is a birational morphism of $\mathcal{C}_m$ to the closure of its image, which is an elliptic curve of type $(2,2)$.

Proof. Analogous to the proof of Lemma 4.1. □

Let $k\in\mathbb{Z}$ be such that $2\le k\le m$ and let $m=m_1+\cdots+m_{k-1}$ be a partition of m as a sum of positive integers. We define a $(k-1)$-*marking of type* $\mathbf{m}=(m_1,\ldots,m_{k-1})$ of an even chain of length $2m$ to be a marking of the $k-1$ lines on R_1

$$f_{1,m_1},f_{1,m_1+m_2},\ldots,f_{1,m_1+m_2+\cdots+m_{k-2}},f_{1,m_1+m_2+\cdots+m_{k-1}}=f_{1,m}$$

and of the $k-1$ lines on R_2

$$f_{2,m+1-m_1},f_{2,m+1-m_1-m_2},\ldots,f_{2,m+1-m_1-\ldots-m_{k-2}},f_{2,m+1-m_1-\ldots-m_{k-1}}=f_{2,1}.$$

Assume that the chain is contained in a nodal hyperplane section H of R. Let Γ_1 and Γ_2 be the sections on R_1 and R_2 contained in H. Figure 3 shows a 3-marked even chain of length 10 of type $(2,1,2)$. The marked lines are thickened. All 10 intersection points between the chain and E are marked with a box, the two distinguished points are marked with filled boxes. In the stable model of the normalization of H at the associated nodes, all lines but the marked ones are contracted (see Examples 4.3 and 4.4 below).

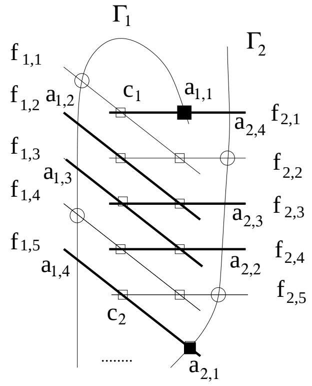

FIGURE 3.

The chain intersects Γ_1 and Γ_2 in a total of $2m$ points evenly distributed between R_1 and R_2. On each Γ_i there are $k-1$ of these intersection points lying on the marked lines, plus the distinguished point $a_{i,1}$. We denote the points lying on $f_{1,m_1+\cdots+m_{j-1}}$ and on $f_{2,m+1-m_1-\cdots-m_{j-1}}$ by $a_{1,j}$ and $a_{2,j}$ respectively, for $j=2,\ldots,k$. We call $a_{i,1}+\cdots+a_{i,k}$ the *k-cycle on Γ_i associated to the chain.* The remaining $2(m-k+1)$ intersection points of the chain with Γ_1 and Γ_2 will be called *the nodes of H associated to the chain.* In Figure 3, the associated nodes are marked with circles.

Let $f_i : E \to \Gamma_i \simeq \mathbb{P}^1$ be the morphism determined by the linear series $|L_i|$, for $i=1,2$. For every pair of points in the k-cycle associated to the chain, we have

$$a_{i,j}+a_{i,j+l} \in f_{i*}|m_{j,l}L_{3-i}-(m_{j,l}-1)L_i|, \quad \text{where } m_{j,l} := m_j+\cdots+m_{j+l-1}. \tag{15}$$

We call the vector $\mathbf{v_m}$ of length $\binom{k}{2}$ of the (lexicographically ordered) integers $m_{j,l}$, the *characteristic vector of the chain.* Note that $m_1,\ldots,m_{k-1}$ appear as coordinates of $\mathbf{v_m}$, gotten for $l=1$, and any other coordinate is strictly bigger.

Example 4.3. In Figure 4 we show the stable model of the partial normalization at the associated nodes of a hyperplane section containing an even chain of length 10 with a 1-marking. The only marked lines are $f_{2,1}$ and $f_{1,5}$.

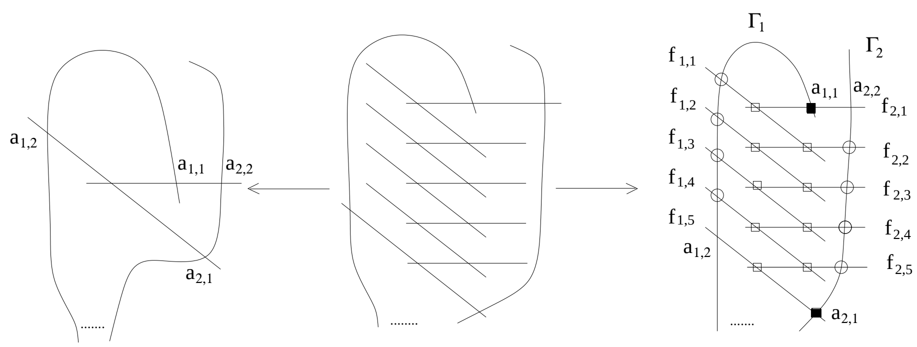

FIGURE 4.

Let X be the curve that is the union of the chain and Γ_1 and Γ_2. The partial normalization at the associated nodes and at $\Gamma_1 \cap \Gamma_2$ is shown on the left of Figure 5 and is stably equivalent to the middle curve in that figure, obtained by adding the vertical $\mathbb{P}^1$. This curve is the domain of a $2:1$ admissible cover onto a stable *pointed* curve of genus zero, where the marked points on the target correspond to the branch points. The same holds for any 1-marking.

Example 4.4. Consider a nodal hyperplane section H of R containing a 5-marked even chain of length 16 (i.e., $k=6$ and $m=8$) of type $(1,2,1,3,1)$, as shown in Figure 6 where the marked lines are thickened. The $\delta = 2(m-k+1) = 6$ marked nodes are the intersection points between Γ_i, $i=1,2$, and the unmarked lines.

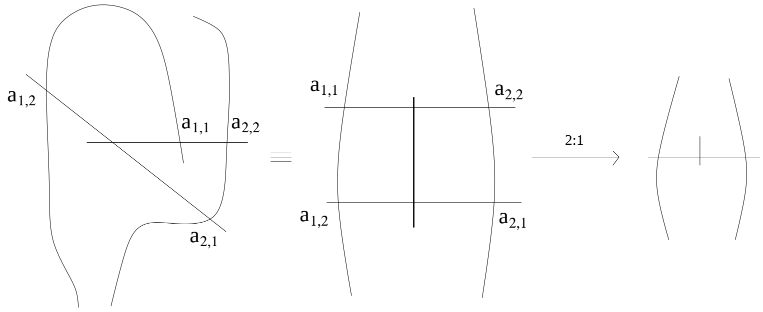

FIGURE 5.

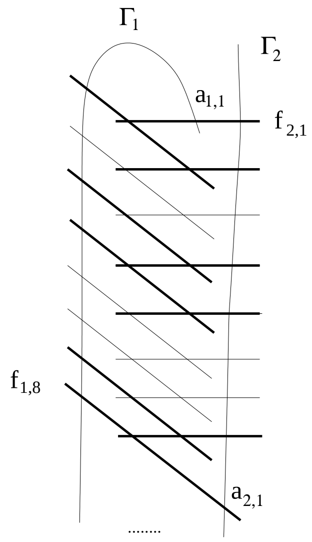

FIGURE 6.

Consider the curve X that is the union of the chain and Γ_1 and Γ_2. The normalization of X at the marked nodes and at $\Gamma_1 \cap \Gamma_2$ has a stable model $\overline{X}$ that looks like the curve on the left in Figure 7. The surviving lines are the marked ones, and there are two lines intersecting both Γ_1 and Γ_2,

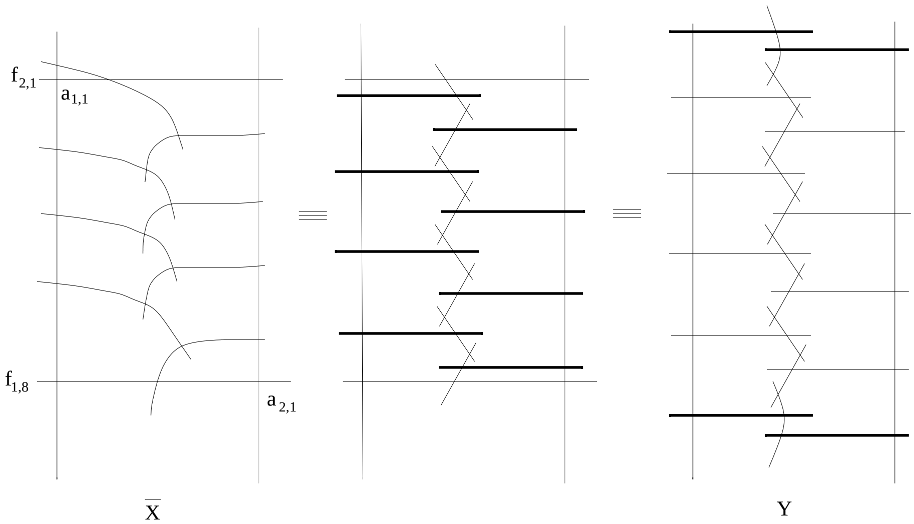

FIGURE 7.

namely the distinguished lines.

In Figure 7 we show a two step construction of a curve Y that is stably equivalent to $\overline{X}$: the curve Y is obtained from $\overline{X}$ by replacing all the nodes of $\overline{X}$ on Γ_1 and Γ_2 with a rational curve connecting the two branches. The additional rational curves intersecting the rest in two points are thickened.

Let g denote the genus of the stable curve Y. We claim that Y lies in $\overline{\mathcal{M}^1_{g,6}}$, equivalently Y is stably equivalent to a curve which is the domain of a degree 6 admissible cover onto a stable pointed genus zero curve. Instead of constructing this curve, we give an indirect proof that Y lies in $\overline{\mathcal{M}^1_{g,6}}$. As shown in Figure 8, the curve $Y = Y_0$ can be deformed in a one-parameter family to a curve Y_t where we have smoothed the four nodes of the four intermediate 2-chains in Y, obtaining the four intermediate $\mathbb{P}^1$'s on Y_t (that are the second to fifth curved curves in the picture of Y_t). Each of these intersects the residual in four points, and we can choose the deformation so that these four-tuples of points are all projective. To see this, consider an abstract curve Y_t by picking Γ_1 and Γ_2 each with its 6 marked points (namely the intersection points with the residual curve) and attach $\mathbb{P}^1$'s at these marked points so as to obtain a configuration like the one appearing in the picture of Y_t without the curved $\mathbb{P}^1$'s; finally we can attach the remaining (curved) $\mathbb{P}^1$'s so as to obtain the final configuration with the required property. Then degenerate Y_t by keeping all curves and their intersections fixed, except for breaking the four intermediate curved $\mathbb{P}^1$'s in two so as to obtain a curve with a configuration like in Y. Any such curve is stably equivalent to Y.

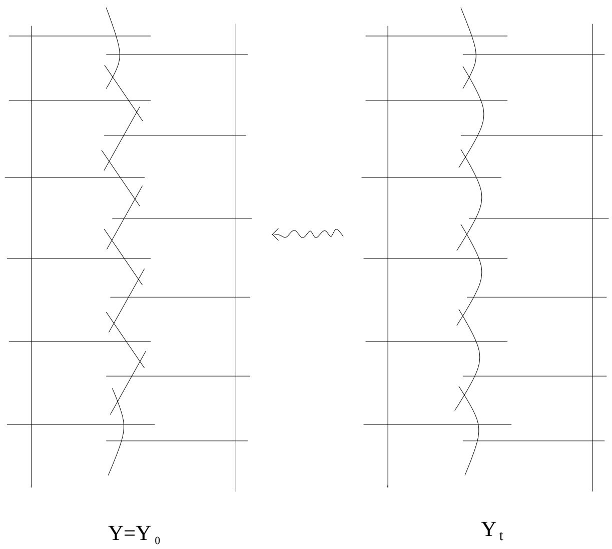

FIGURE 8.

Next we prove that Y_t is stably equivalent to a curve with a degree 6 admissible cover onto a stable pointed genus zero curve. The curve Y_t has two extremal curved $\mathbb{P}^1$'s intersecting the rest in three points. Add two rational tails each intersecting one of these $\mathbb{P}^1$'s, so as to make the four nodes on them projective to the four nodes on each of the remaining curved $\mathbb{P}^1$'s. The resulting curve is the second one on the left in Figure 9, where the added curves are thickened. Then we resolve the five intersection points of the curved $\mathbb{P}^1$'s and add five $\mathbb{P}^1$'s joining the two branches over each node. The resulting curve is the third one on the left in Figure 9, where the added curves are thickened. This curve is a degree 6 admissible cover of a stable pointed genus zero curve, as shown on the right in Figure 9.

In conclusion, we proved that Y_t lies in $\overline{\mathcal{M}^1_{g,6}}$, whence also $Y = Y_0$ does.

A few facts observed in the last example hold for any nodal hyperplane section H containing an even chain of length $2m$ with a $(k-1)$-marking:

- the marked lines are the only ones surviving in the stable model $\overline{X}$ of the partial normalization at the associated nodes of the curve X that is the union of the chain and Γ_1 and Γ_2;
- in $\overline{X}$ each nondistinguished marked line intersects only two other marked lines in two distinct points and only one of Γ_1 and Γ_2, and the set of nondistinguished marked lines on the same R_i are disjoint;

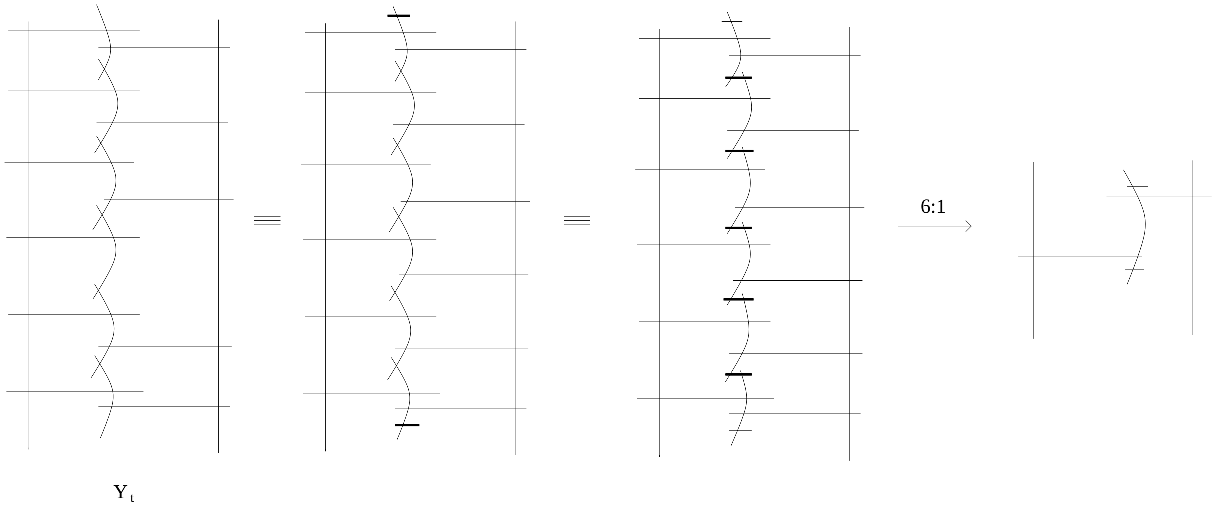

FIGURE 9.

- in $\overline{X}$ the two distinguished marked lines intersect only one other marked line and both Γ_1 and Γ_2.
- which lines intersect which in the stable model is a property of the marking and not of the hyperplane section containing it.

In particular, $\overline{X}$ looks like one of the two curves in Figure 10.

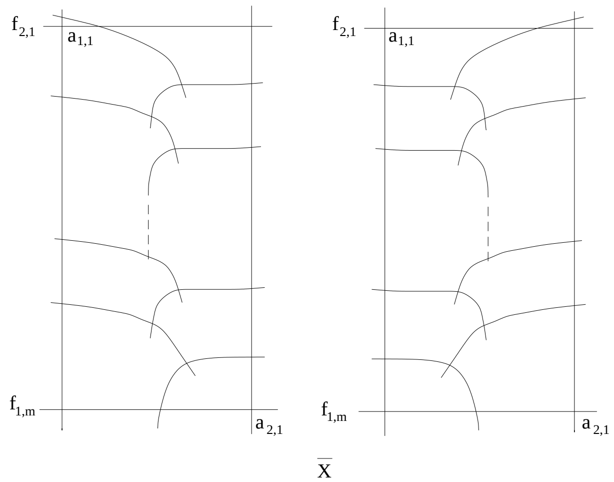

FIGURE 10.

Lemma 4.5. *Let H be any nodal hyperplane section of R containing an even chain with a $(k-1)$-marking. Let Γ_1 and Γ_2 be the sections on R_1 and R_2 contained in H and consider the curve X union of the chain and Γ_1 and Γ_2. Then the stable model of the partial normalization of X at the associated nodes and at $\Gamma_1 \cap \Gamma_2$ lies in $\overline{\mathcal{M}^1_{g,k}}$, where g is the genus of X.*

Proof. Let $\overline{X}$ be the stable model of X, which looks like one of the two curves in Figure 10. Performing the same steps as in Figure 7, namely replacing all the nodes on Γ_1 and Γ_2 with a $\mathbb{P}^1$ connecting the two branches, we see that $\overline{X}$ is stably equivalent to the curve Y shown on the left in Figure 11.

By the same argument as in Example 4.4, the curve $Y = Y_0$ can be deformed in a one-parameter family where the general curve Y_t looks like the second curve on the left in Figure 11 and has the following property: Each of the second to the last but one curved $\mathbb{P}^1$ intersects the residual in four points, and all these four-tuples of points are projective. The latter is stably equivalent to a curve admitting a degree k admissible cover onto a stable pointed genus zero curve, as shown on the right hand side of Figure 11. □

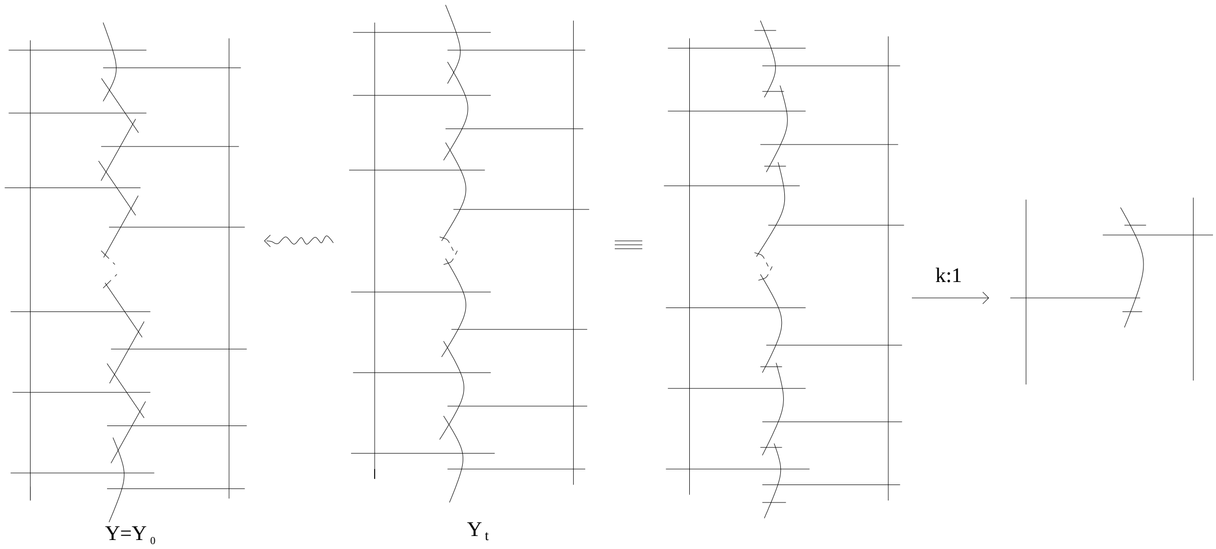

FIGURE 11.

Remark 4.6. In the above degree k admissible cover, the branch points lie on the images of Γ_1 and Γ_2 and on the two horizontal tails. The latter can be chosen to be simple.

Remark 4.7. If the chains do not pass through any of the 16 double points of $\mathcal{X}$, we can consider them as chains of rational curves on S_0. By Remark 1.6, given finitely many chains as above, there is a degeneration as in § 1.5 such that none of the 16 double points of $\mathcal{X}$ lie on any chain.

5. LIMITS OF NODAL CURVES

Fix two integers $p, k \geq 2$, set $n := \lfloor p/2 \rfloor$ and $\epsilon := p - 2n$. For any integer j satisfying $1 \leq j \leq n$, define three integers $\alpha_{1,j}$, $\alpha_{2,j}$, β_j and the vector $\mathbf{x} := (\alpha_{1,1}, \alpha_{2,1}, \beta_1, \ldots, \alpha_{1,n}, \alpha_{2,n}, \beta_n)$. We fix once and for all one of the 16 double points of $\mathcal{X}$, which we denote by ξ.

Definition 5.1. We define $V(\mathbf{x})$ to be the locally closed subset of $|\mathcal{O}_R(1)|$ consisting of nodal curves C containing excatly $\alpha_{1,j}$ odd chains of length $2j-1$ based on R_1, $\alpha_{2,j}$ odd chains of length $2j-1$ based on R_2 and β_j even chains of length $2j$, such that no chain contains any of the 16 double points of $\mathcal{X}$ (cf. Remark 4.7). Moreover, denoting by Γ_1 and Γ_2 the sections on the two scrolls residual to the chains in C, we require that $\Gamma_1 \cap \Gamma_2$ is disjoint from the singular locus of $\mathcal{X}$ if p is even, whereas $\Gamma_1 \cap \Gamma_2$ intersects the singular locus of $\mathcal{X}$ at ξ, if p is odd.

Each curve in the definition comes naturally equipped with a subscheme of

$$d = d(\mathbf{x}) := \sum_{j=1}^{n} \Big((2j-1)(\alpha_{1,j} + \alpha_{2,j}) + 2j\beta_j \Big) + \epsilon, \tag{16}$$

nodes consisting of *all* the nodes lying in the smooth locus of R, plus ξ when p is odd.

By abuse of notation, we also denote by $V(\mathbf{x})$ the subset of $|\mathcal{H}_0|$ on S_0 consisting of the total transforms of the above curves. The difference is that in the case $\epsilon = 1$, the total transform of Γ_2 contains the exceptional divisor lying over ξ, and the corresponding node is the intersection point of the strict transform of Γ_2 with this exceptional divisor, hence it also lies in the smooth locus of S_0.

Proposition 5.2. *Assume that*

$$\sum_{j=1}^{n} \Big[2j(\alpha_{1,j} + \alpha_{2,j}) + (2j+1)\beta_j \Big] + \epsilon \leq p + 1. \tag{17}$$

Then

(i) $V(\mathbf{x})$ *is a smooth component of dimension* $p - d$ *of* $V_{|\mathcal{O}_{S_0}(1)|,d}(S_0)$*;*
(ii) $V(\mathbf{x})$ *is a component of the flat limit of* $V_{|H|,d}(S)$ *with* $(S, H) \in \mathcal{K}_p$ *general.*

Proof. By Lemmas 4.1 and 4.2, we have a morphism

$$g : V(\mathbf{x}) \longrightarrow \prod_{i=1}^{2}\prod_{j=1}^{n} \mathrm{Sym}^{\alpha_{i,j}} |jL_i - (j-1)L_{3-i}| \times \prod_{j=1}^{n} G_j^{\beta_j}.$$

Take a general point η of the target, which has dimension

$$t := \sum_{j=1}^{n} (\alpha_{1,j} + \alpha_{2,j} + \beta_j). \tag{18}$$

By Lemmas 4.1 and 4.2, each coordinate of η uniquely defines a chain. The reduced intersection between the union of these chains and E is an effective divisor. We denote by D_η the sum of this divisor and $\epsilon\xi$, which has degree s equal to the left hand side of (17).

The fibre of g over $\eta \in \mathrm{Im}(g)$ is a dense open subset of a projective space of dimension

$$f := \dim |\mathcal{O}_{S_0}(1) - C_\eta| = \dim |\mathcal{O}_E(1) - D_\eta|. \tag{19}$$

If $s \leq p$ then $f = p - s \geq 0$ so that g is surjective. Hence $V(\mathbf{x})$ is irreducible of dimension

$$t + f = \sum_{j=1}^{n} (\alpha_{1,j} + \alpha_{2,j} + \beta_j) + p - \sum_{j=1}^{n} \Big[2j(\alpha_{1,j} + \alpha_{2,j}) + (2j+1)\beta_j - \epsilon \Big] = p - d,$$

by (16), (18) and (19). If $s = p + 1$, then g is not surjective and its image is the codimension one subspace of the target consisting of all η such that $D_\eta \in |\mathcal{O}_{S_0}(1)|$. Hence the dimension of any fiber of g is zero, so that, by (16) and (18), the variety $V(\mathbf{x})$ is irreducible of dimension

$$t - 1 = \sum_{j=1}^{n} (\alpha_{1,j} + \alpha_{2,j} + \beta_j) - 1 = p - d.$$

Since the d nodes are not disconnecting (cf. § 4), the tangent space to $V_{|\mathcal{O}_{S_0}(1)|,d}(S_0)$ at a point of $V(\mathbf{x})$ has dimension $p - d$. Hence (i) follows.

Assertion (ii) follows by Lemma 1.3. □

6. Limits of k-gonal nodal curves

Let $k \geq 2$ be an integer. We henceforth restrict ourselves to studying curves in $V(\mathbf{x})$ with only one even chain, that is, $\beta_j = 1$ for one j and $\beta_j = 0$ for all other j. Let e be the index for which $\beta_e = 1$, and set $\mathbf{a}_i := (\alpha_{i,1}, \ldots, \alpha_{i,n})$. We will use the notation $V(\mathbf{a}_1, \mathbf{a}_2, e)$ for $V(\mathbf{x})$. In this case (16) reads

$$d = d(\mathbf{a}_1, \mathbf{a}_2, e) := \sum_{j=1}^{n} \Big((2j-1)(\alpha_{1,j} + \alpha_{2,j}) \Big) + 2e + \epsilon.$$

6.1. k-gonal nodal curves in the central fibre.

Definition 6.1. Let $e = e_1 + \ldots + e_{k-1}$ be a partition of e in positive integers, and set $\mathbf{e} = (e_1, \ldots, e_{k-1})$. We define $V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{e})$ to be the locally closed subset of $V(\mathbf{a}_1, \mathbf{a}_2, e)$ consisting of curves C such that:

- the even chain of length $2e$ contained in C has a $(k-1)$-marking of type $\mathbf{e}$;
- $\Gamma_1 \cap \Gamma_2 = \emptyset$ if p is even and $\Gamma_1 \cap \Gamma_2 = \{\xi\}$ if p is odd;
- for $i = 1, 2$, each 2-cycle on Γ_i associated to an odd chain and the k-cycle associated to the even chain all belong to divisors of the same g^1_k on Γ_i.

The number of the associated nodes of the chains in C, plus the one in ξ if p is odd, is

$$\delta = \delta(\mathbf{a}_1, \mathbf{a}_2, e) := \sum_{j=1}^{n} \Big((2j-1)(\alpha_{1,j} + \alpha_{2,j}) \Big) + 2(e - k + 1) + \epsilon = d - 2(k-1). \tag{20}$$

We call these the *marked nodes of* C. The number δ does not depend on $\mathbf{e}$, but the δ nodes do depend on the $(k-1)$-marking.

We define the integers

$$\hat{\alpha}_j = \#\{\text{coordinates of } \mathbf{v}_{\mathbf{e}} \text{ equal to } j\}, \tag{21}$$

where $\mathbf{v_e}$ is the characteristic vector of the even chain in C (see p. 11). The importance of $\mathbf{v_e}$ and of the set of integers $\hat{\alpha}_j$ is the following: as we saw in § 4.2, there is a one-to-one correspondence between the set of 2-subcycles of the k-cycle associated to the even chain on Γ_i and the coordinates of $\mathbf{v_e}$, and, by (15), the coordinate a of $\mathbf{v_e}$ corresponding to a 2-subcycle Z tells us that

$$Z \in f_{i*}|aL_{3-i} - (a-1)L_i|. \tag{22}$$

Recall that the $k-1$ smallest coordinates in $\mathbf{v_e}$ are the ones appearing as coordinates of $\mathbf{e}$ (see 11).

Lemma 6.2. *If C is a curve in $V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{e})$, the stable model of its partial normalization at its δ marked nodes lies in $\overline{\mathcal{M}^1_{p-\delta,k}}$, hence*

$$V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{e}) \subseteq \overline{V^k_{|H_0|,\delta}(S_0)}.$$

Proof. Let $\widetilde{C}$ be the partial normalization of C. We use the same notation for the components of C and $\widetilde{C}$. In particular, $\widetilde{C}$ contains the two smooth rational curves Γ_1 and Γ_2 as in Definition 5.1. The stable model $\overline{C}$ of $\widetilde{C}$ has genus $g = p-\delta$. It contains two rational curves $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ with $\sum_j \alpha_{2,j}$ and $\sum_j \alpha_{1,j}$ nodes, respectively (coming from identifying the distinguished pairs on each odd chain), and intersecting transversally in ξ if p is odd, not intersecting at all if p is even. In addition to $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$, the curve $\overline{C}$ contains a chain of $2(k-1)$ smooth rational curves coming from the marked lines of the even chain, looking as in Figure 10. The chain intersects each $\overline{\Gamma}_i$ transversally in k points, correponding to the k-cycle on Γ_i associated to the even chain.

The data yields that each $\overline{\Gamma}_i$ admits a $k:1$ map to $\mathbb{P}^1$ identifying the k intersection points with the chain. The result follows from Lemma 4.5. □

Our objective is to prove that, under suitable conditions, $V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{e})$ is nonempty of dimension $2(k-1)$ and fills one or more components of $\overline{V^k_{|H_0|,\delta}(S_0)}$. To do so, we need some intermediate results.

6.2. **Some technical results.** Recall the maps $f_i : E \to \Gamma_i \simeq \mathbb{P}^1$ determined by $|L_i|$, $i = 1, 2$. We have the induced maps $f_i^{(2)} : \mathrm{Sym}^2(E) \to \mathrm{Sym}^2(\mathbb{P}^1)$.

As customary, we identify $\mathrm{Sym}^2(\mathbb{P}^1)$ with $\mathbb{P}^2$: fix an irreducible conic $\Delta \simeq \mathbb{P}^1$, and identify a divisor $x+y$ of Δ with:

- the pole of the line $\langle x, y\rangle$ with respect to Δ, if $x \neq y$;
- the point $x \in \Delta$, if $x = y$.

In this way Δ is identified with the *diagonal* of $\mathrm{Sym}^2(\mathbb{P}^1)$ and the *coordinate curve* $\{x+y, y \in \mathbb{P}^1\}$ with the tangent line ℓ_x to Δ at x.

We denote by $\mathfrak{c}_{i,j}$, for $i = 1, 2$ and $1 \le j \le n$, the image via $f_i^{(2)}$ of the smooth rational curves in $\mathrm{Sym}^2(E)$ defined by the pencils $|jL_{3-i} - (j-1)L_i|$. These are distinct conics, each intersecting Δ in four distinct points corresponding to the ramification points of the pencils.

Consider $\mathbb{Q} = \mathbb{P}^1 \times \mathbb{P}^1$ with the two projections $\pi_i : \mathbb{Q} \to \mathbb{P}^1$, $i = 1, 2$. Fix a positive integer k and look at the line bundle $\mathcal{O}_{\mathbb{Q}}(k,k) := p_1^*(\mathcal{O}_{\mathbb{P}^1}(k)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(k))$, whose space of sections is $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k))^{\otimes 2}$. The two subspaces $\mathrm{Sym}^2(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)))$ and $\wedge^2 H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k))$ are invariant (resp. anti-invariant) under the natural involution that exchanges the coordinates. Hence they are pull-backs of sections of line bundles, $\mathcal{O}_k^+$ and $\mathcal{O}_k^-$ respectively, on $\mathrm{Sym}^2(\mathbb{P}^1)$.

Let us focus on $\mathcal{O}_k^-$. One has

$$H^0(\mathrm{Sym}^2(\mathbb{P}^1), \mathcal{O}_k^-) \simeq \wedge^2 H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) \simeq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-1)).$$

Therefore the linear system $|\mathcal{O}_k^-|$ identifies with $|\mathcal{O}_{\mathbb{P}^2}(k-1))|$ and also with $\mathbb{P}(\wedge^2 H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)))$. Under the former isomorphism, a point $\mathfrak{g}$ of the grassmannian $\mathbb{G}(1,k) \subset \mathbb{P}(\wedge^2 H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)))$, which can be identified with a linear series g^1_k on $\mathbb{P}^1$, corresponds to the degree $k-1$ curve in $\mathbb{P}^2$

$$C_{\mathfrak{g}} = \{W \in \mathrm{Sym}^2(\mathbb{P}^1) \mid \mathfrak{g}(-W) \ge 0\}.$$

The family of curves $\{C_{\mathfrak{g}}\}_{\mathfrak{g} \in \mathbb{G}(1,k)}$ has dimension $2(k-1) = \dim(\mathbb{G}(1,k))$.

Lemma 6.3. *For general choices of the line bundles L_i, $i = 1, 2$, we have:*

(i) *no curve $\mathfrak{c}_{i,j}$ is contained in a curve $C_{\mathfrak{g}}$, for $\mathfrak{g} = g^1_k$ on $\mathbb{P}^1$;*

(ii) *given any reduced effective divisor D of degree k on $\mathbb{P}^1$ and a general $\mathfrak{g}$ containing D, the curve $C_{\mathfrak{g}}$ intersects each $\mathfrak{c}_{i,j}$ transversally in $2(k-1)$ distinct points;*
(iii) *in addition, none of these $2(k-1)$ points is fixed varying $\mathfrak{g}$.*

Proof. By moving L_{3-i}, the conic $\mathfrak{c}_{i,j}$ moves in $\mathbb{P}^2$ in a 1-dimensional family containing the diagonal Δ (obtained for $L_1 = L_2$). But Δ is not contained in any $C_{\mathfrak{g}}$, proving (i). Similarly, it suffices to prove (ii) for the intersection with Δ of a $C_{\mathfrak{g}}$, with $\mathfrak{g}$ general containing D, which is obvious, since the intersection points correspond to the ramification points of $\mathfrak{g}$. Part (iii) easily follows. □

In the following lemma, whose proof is left to the reader, we use the notation introduced in § 4.2.

Lemma 6.4. *Lemma 4.1 can be rephrased as saying that an odd chain of length $2m-1$ based on R_i determines and is determined by a point on the conic $\mathfrak{c}_{3-i,m}$. Similarly, Lemma 4.2(b) can be rephrased as saying that an even chain of length $2m$ determines a unique point in $\mathfrak{c}_{1,m} \times \mathfrak{c}_{2,m}$.*

6.3. **Nonemptiness and dimension of limit k-gonal nodal curves systems.** Consider the following conditions:

$$\sum_{j=1}^{n} \Big[2j(\alpha_{1,j} + \alpha_{2,j})\Big] + 2e + \epsilon = p \tag{23}$$

$$\alpha_{i,j} + \hat{\alpha}_j \leq 2(k-1) \ \text{ for all } i, j. \tag{24}$$

Proposition 6.5. *Under* (23)-(24)*, the variety $V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{e})$ is nonempty of dimension $2(k-1)$ and is a component of $\overline{V^k_{|H_0|,\delta}(S_0)}$.*

Proof. Under (23), the condition on $\Gamma_1 \cap \Gamma_2$ in Definition 6.1 is satisfied and any element C in $V(\mathbf{a}_1, \mathbf{a}_2, e)$ cuts on E a divisor of the linear system

$$\Big|\sum j(\alpha_{1,j}L_1 + \alpha_{2,j}L_2) + eL_i + a_{i,1} + \epsilon\xi\Big|, \quad i = 1, 2.$$

This system coincides with $|\mathcal{O}_E(1)|$, and this determines uniquely $a_{1,1}$, or equivalently $a_{2,1}$, hence it uniquely determines the even chain contained in C. This even chain actually exists, i.e. no involved line is tangent to E: otherwise $\mathcal{O}_E(2a_{1,1})$ would be a linear combination of L_1 and L_2 in the Picard group of E, hence $\mathcal{O}_E(2)$ would be a linear combination of L_1 and L_2, against the generality assumption. In conclusion, all curves in $V(\mathbf{a}_1, \mathbf{a}_2, e)$ contain the same even chain. Given the marking $\mathbf{e}$, the two k-cycles $a_{i,1} + \cdots + a_{i,k}$, $i = 1, 2$, associated to this chain are uniquely determined.

Consider the set of curves $C_{\mathfrak{g}_i}$ in $\mathbb{P}^2 \simeq \mathrm{Sym}^2(\Gamma_i)$ corresponding to $\mathfrak{g}_i = g^1_k$ on Γ_i containing $a_{i,1} + \cdots + a_{i,k}$. This family of g^1_k's is a $\mathbb{P}^{k-1}$. Each curve $C_{\mathfrak{g}_i}$ passes through the $\binom{k}{2}$ points corresponding to subcycles of length two of $a_{i,1} + \cdots + a_{i,k}$. By (21) and (22) we have that $\hat{\alpha}_j$ of these points lie on the conic $\mathfrak{c}_{i,j}$. A general $C_{\mathfrak{g}_i}$ intersects each $\mathfrak{c}_{i,j}$ in $2(k-1)$ distinct points by Lemma 6.3(ii). Due to (24), we can pick $\alpha_{i,j}$ of them not coinciding with any of the $\hat{\alpha}_j$ aforementioned points. By Lemma 6.3(iii), each of these $\alpha_{i,j}$ intersection points gives rise to an odd chain of length $2j-1$ with distinguished pair given by the 2-cycle corresponding to the point itself. Then the construction in (the proof of) Proposition 5.2 implies the existence of curves in $V(\mathbf{a}_1, \mathbf{a}_2, e)$ containing all the above chains. These curves, by construction, actually lie in $V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{e})$. This shows that $V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{e})$ is nonempty, birational to $\mathbb{P}^{k-1} \times \mathbb{P}^{k-1}$, hence it has dimension $2(k-1)$.

We have left to prove that $V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{e})$ is a component of $\overline{V^k_{|H_0|,\delta}(S_0)}$.

Assume that there is a flat family of curves parametrized by a disk in $\overline{V_{|H_0|,\delta}(S_0)}$ with central fibre a general curve C_0 in $V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{e})$ and general fibre C_t not lying in $V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{e})$. Then at least one of the nodes among $a_{i,2}, \ldots, a_{i,k}$, $i = 1, 2$, is smoothed. Suppose this happens for $i = 1$. Then, in the stable reduction of the partial normalization of C_t, the curve tending to Γ_1 (i.e., the irreducible section on R_1 that is a component of C_t) intersects the residual curve in more than k points and the residual curve is connected. Such a curve cannot be stably equivalent to a curve with an admissible cover of degree k onto a stable pointed genus zero curve. Hence $C_t \notin \overline{V^k_{|H_0|,\delta}(S_0)}$, and it follows that $V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{e})$ is a component of $\overline{V^k_{|H_0|,\delta}(S_0)}$. □

Remark 6.6. Propositions 1.5 and 6.5 imply that the curves in $V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{e})$ deform to nodal curves in $\overline{V^k_{|H|,\delta}(S)}$ on the general $(S, H) \in \mathcal{K}_p$. The $k-1$ nodes $a_{i,j}$, $1 \leq j \leq k-1$, on each Γ_i, $i = 1, 2$, must smooth. Otherwise the general curve in a component of $\overline{V^k_{|H|,\delta}(S)}$ would have some neutral node tending to one of the above. This is not possible since $a_{i,1} + \cdots + a_{i,k}$ is a fibre of the admissible cover of the curve in $V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{e})$.

It follows from the proof of Proposition 6.5 that the g^1_k on the normalization of the general curve in $V^k_{|H|,\delta}(S)$ has simple ramification. Indeed, the limit ramification points are the ones described in Remark 4.6 and the ones tending to the nodes of $\overline{\Gamma}_i$, $i = 1, 2$. The former ramification is simple, because the ramification of the general g^1_k on Γ_i containing $a_{i,1} + \cdots + a_{i,k}$, $i = 1, 2$, is simple. The latter is simple because, in the admissible cover, each node is replaced by a $\mathbb{P}^1$ joining the two branches, which goes $2:1$ to a $\mathbb{P}^1$, hence the ramification is simple there.

Finally, all δ noes on the general curve in $V^k_{|H|,\delta}(S)$ are non-neutral with respect to the g^1_k, as this is the case for the δ marked nodes on the limit curves in $V(\mathbf{a}_1, \mathbf{a}_2, \mathbf{e})$.

7. The proof of the main theorem

In this section we prove Theorem 0.1. In the range $g \geq 2(k-1)$ where nonemptiness of $V^k_{|H|,\delta}$ is trivial, we have left to prove the following:

Lemma 7.1. *Under the same assumptions as in Theorem 0.1, assume that $\delta \geq p - 2(k-1)$. Then there is a curve in $V_{|H|,\delta}$ with only non-neutral nodes with respect to a g^1_k, with simple ramification, on its normalization.*

Proof. It suffices to find the desired limit curve on R. Let $\mathbf{x}$ be the vector where the only nonzero coordinates are $\alpha_{1,1} = g := p - \delta$ and $\beta_e = 1$, with $e = \delta - \frac{p+\epsilon}{2}$. By Proposition 5.2, the variety $V(\mathbf{x})$ is a smooth component of dimension g of $V_{|\mathcal{O}_{S_0}(1)|,\delta}(S_0)$ and is a component of the flat limit of $V_{|H|,\delta}(S)$ with $(S, H) \in \mathcal{K}_p$ general.

Besides the g odd chains of length one based on R_1 and the even chain of length $2e$, a curve C in $V(\mathbf{x})$ contains the residual sections Γ_i on R_i, $i = 1, 2$, which intersect transversally in ξ if $\epsilon = 1$ and are disjoint if $\epsilon = 0$. In the stable model of the normalization of C at its δ nodes lying in the smooth locus of S_0, the curve Γ_1 gets contracted and we are left with the image $\overline{\Gamma}_2$ of Γ_2, where each distinguished pair of points is identified to a node; in total there are g such nodes. Since $g \leq 2(k-1)$, there is a g^1_k on Γ_2 identifying these pairs of points, and for general choices of the of the odd chains, each such point lies outside of the ramification and no two pairs of points form part of the same divisor in the g^1_k. The assertions about the δ nodes of C being non-neutral and the ramification of the admissible cover being simple follow from the above considerations. $\square$

We are left to prove Theorem 0.1 in the range $\delta < p - 2(k-1)$. The result follows by Propositions 6.5 and 1.5, Remark 6.6 and the next result.

Proposition 7.2. *Let p, k, δ be integers satisfying $p \geq 2$, $k \geq 2$, $\delta \leq p - 2(k-1)$ and (2). Then there are integers $\alpha_{i,j}$ and e_i satisfying (23)-(24) with $\delta(\mathbf{a}_1, \mathbf{a}_2, e) = \delta$.*

To prove this, we need two auxiliary results.

Lemma 7.3. *Assume there are integers $\alpha_{i,j}$ and e_i satisfying (23)-(24) with $\delta_0 = \delta(\mathbf{a}_1, \mathbf{a}_2, e)$. Then, for any integer δ' satisfying*

$$\delta_0 \leq \delta' \leq p - 2(k-1)$$

there are integers $\alpha'_{i,j}$ and e'_i satisfying (23)-(24) with $\delta(\mathbf{a}'_1, \mathbf{a}'_2, e') = \delta'$.

Proof. Assume without loss of generality that $e_1 \leq e_2 \leq \cdots \leq e_{k-1}$. If all $\alpha_{i,j} = 0$, there is nothing to prove, because $p = 2e + \epsilon$ by (23) and $\delta_0 = 2(e - k + 1) + \epsilon = p - 2(k-1)$ by (20). We may therefore assume that not all $\alpha_{i,j} = 0$. To prove the lemma, it suffices to find integers $\alpha'_{i,j}$ and e'_i satisfying the conditions (23)-(24) with $\delta(\mathbf{a'}_1, \mathbf{a'}_2, e') = \delta_0 + 1$, where $e' = e'_1 + \cdots + e'_{k-1}$. For this, take $\alpha_{s,t} \neq 0$ and define

$$\alpha'_{i,j} = \begin{cases} 0 & \text{if } (i,j) = (s,t), \\ \alpha_{i,j} & \text{if } (i,j) \neq (s,t), \end{cases} \quad \text{and} \quad e'_i = \begin{cases} e_i & \text{if } i = 1, \ldots, k-2, \\ e_{k-1} + \alpha_{s,t} & \text{if } i = k-1. \end{cases}$$

These integers satisfy the desired conditions. □

Define now
$$m := \max\{n \in \mathbb{Z} \mid 4(k-1)n(n+1) \leq p\}$$
and
$$t := \max\{n \in \mathbb{Z} \mid 4(k-1)m(m+1) + 2(m+1)t \leq p\}, \text{ i.e. } t = \left\lfloor \frac{p}{2(m+1)} \right\rfloor - 2m(k-1).$$
We have a unique representation
$$p = 4(k-1)m(m+1) + 2(m+1)t + \lambda + \epsilon. \tag{25}$$
Note that
$$0 \leq t < 4(k-1) \ \text{ and } \ 0 \leq \lambda \leq 2m, \ \text{ with } \lambda \text{ even.} \tag{26}$$

Lemma 7.4. *Given p and k, the minimal integer δ satisfying* (2) *is*
$$\delta_0 = (4m^2-1)(k-1) + (2m+1)t + \lambda + \epsilon + l_0,$$
where
$$l_0 = \begin{cases} \left\lceil \frac{k-1-\lambda-t}{2m+1} \right\rceil & \text{if } \lambda + t < k-1, \\ \left\lceil \frac{t-3(k-1)-\lambda}{2m+3} \right\rceil & \text{if } \lambda + 3(k-1) < t, \\ 0, & \text{in all other cases.} \end{cases} \tag{27}$$

Proof. Straightforward computation. □

Proof of Proposition 7.2. We will find integers $\alpha_{i,j}$ and e_i satisfying (23)-(24) with $\delta(\mathbf{a}_1, \mathbf{a}_2, e) = \delta_0$ as in Lemma 7.4. The result then follows from Lemma 7.3. We divide the treatment in four cases.

• **The case** $k-1 \leq t \leq \lambda + 3(k-1)$**.** Set $\lambda_1 := \min\{\lambda/2, k-1\}$ and
$$e_1 = m + 2 + \frac{\lambda}{2} - \lambda_1, \ e_2 = \ldots = e_{\lambda_1} = m+2, \ e_{\lambda_1+1} = \cdots = e_{k-1} = m+1.$$
Then $e := \sum e_i = (k-1)(m+1) + \frac{\lambda}{2}$. We set
$$\alpha_{1,j} = \alpha_{2,j} = 2(k-1), \ j = 1, \ldots, m,$$
$$\alpha_{1,m+1} = \left\lceil \frac{t-(k-1)}{2} \right\rceil, \ \alpha_{2,m+1} = \left\lfloor \frac{t-(k-1)}{2} \right\rfloor,$$
$$\alpha_{1,j} = \alpha_{2,j} = 0, \ j > m+1.$$
Then (23)-(24) are verified; the crucial case for the latter is $j = m+1$, where we have
$$\alpha_{i,m+1} + \hat{\alpha}_{m+1} \leq \left\lceil \frac{t-(k-1)}{2} \right\rceil + k - 1 - \lambda_1 = \left\lceil \frac{t+(k-1)-\lambda}{2} \right\rceil \leq \left\lceil \frac{4(k-1)}{2} \right\rceil = 2(k-1),$$
using $t \leq \lambda + 3(k-1)$. By (20), we obtain
$$\delta(\mathbf{a}_1, \mathbf{a}_2, e) = (4m^2-1)(k-1) + (2m+1)t + \lambda + \epsilon = \delta_0.$$

• **The case** $\lambda < t - 3(k-1)$**.** Let l_0 be as in (27). Then l_0 is the minimal integer l such that $\lambda + (2m+3)l \geq t - 3(k-1)$ and $0 < l_0 \leq k-1$. We set $\lambda' := \lambda + 2(m+1)l_0$ and $t' := t - l_0$. Then $\lambda' \geq t' - 3(k-1)$ and $t' > 2(k-1)$. Then we can argue as in the previous case with λ and t substituted by λ' and t', and compute
$$\begin{aligned} \delta(\mathbf{a'}_1, \mathbf{a'}_2, e') &= (4m^2-1)(k-1) + (2m+1)t' + \lambda' + \epsilon \\ &= 4(k-1)m^2 + (2m+1)t + \lambda + \epsilon - (k-1-l_0) = \delta_0. \end{aligned}$$

• **The case** $t < k-1$ **and** $\lambda + t \geq k-1$**.** We must have $m \geq 1$. Indeed, if $m = 0$, then $\lambda = 0$ by (26) and the two inequalities in our assumption cannot both be satisfied.

Set $\lambda_1 := \lfloor (k-t)/2 \rfloor$. Then $\lambda_1 \geq 1$ and we can write $k - 1 = 2\lambda_1 + t - \epsilon_0$, with $\epsilon_0 = 0$ or 1. In particular, $\lambda_1 \leq \lambda/2$. We set
$$e_1 = \cdots = e_{\lambda_1-\epsilon_0} = m, \ e_{\lambda_1-\epsilon_0+1} = \cdots = e_{k-2} = m+1, \ e_{k-1} = m + 1 + \frac{\lambda}{2} - \lambda_1.$$

Then $e := \sum e_i = (k-1)(m+1) + \epsilon_0 + \lambda/2 - 2\lambda_1$. We set

$$\begin{aligned} \alpha_{1,j} = \alpha_{2,j} = 2(k-1),\ j = 1, \dots, m-1, \\ \alpha_{1,m} = 2(k-1) - \lambda_1,\ \alpha_{2,m} = 2(k-1) - \lambda_1 + \epsilon_0 \\ \alpha_{1,j} = \alpha_{2,j} = 0,\ j \geq m+1. \end{aligned}$$

Then (23)-(24) are verified; the crucial case for the latter is $j = m$, where we have

$$\alpha_{i,m} + \hat{\alpha}_m \leq \Big(2(k-1) - \lambda_1 + \epsilon_0\Big) + \Big(\lambda_1 - \epsilon_0\Big) = 2(k-1).$$

By (20), we compute

$$\delta(\mathbf{a}_1, \mathbf{a}_2, e) = (4m^2-1)(k-1) + (2m+1)t + \lambda + \epsilon = \delta_0.$$

• **The case** $\lambda + t < k-1$**.** Again we have $m \geq 1$. Indeed, if $m = 0$ then $\lambda = 0$, hence $p = 2t + \epsilon \leq 2k-3$, thus $\delta \leq p - 2(k-1) \leq -1$, a contradiction.

Let l_0 be as in (27). Then l_0 is the minimal integer l such that $\lambda + (2m+1)l + t \geq k-1$ and $0 < l \leq k-1$. We set $\lambda' := \lambda + 2ml$ and $t' := t + l$. Then $\lambda' + t' \geq k-1$. Moreover, the minimality of l_0 yields $\lambda + (2m+1)(l_0-1) + t < k-1$, thus $t' \leq k-1$. Hence $\lambda'_1 := \lfloor (k-t')/2 \rfloor \geq 0$.

Write $k - 1 = 2\lambda'_1 + t' - \epsilon'_0$, with $\epsilon'_0 = 0$ or 1. In particular, $\lambda'_1 \leq \lambda'/2$. Moreover, as $t' \leq k-1$, we have that $\epsilon'_0 = 0$ if $\lambda'_1 = 0$. Therefore we may argue as in the previous case with λ, t, λ_1 and ϵ_0 substituted by λ', t', λ'_1 and ϵ'_0. We compute

$$\begin{aligned} \delta(\mathbf{a'}_1, \mathbf{a'}_2, e') &= (4m^2-1)(k-1) + (2m+1)t' + \lambda' + \epsilon \\ &= 4(k-1)m^2 + (2m+1)t + \lambda + \epsilon - (k-1-l_0) = \delta. \end{aligned}$$

□

We conclude the section by giving an explicit characterizetion of the genera p where Theorem 0.1 is not optimal, i.e., the cases where equality is attained in (1) and p is odd (so that $\epsilon = 1$).

Proposition 7.5. *Assume p is odd and $k \geq 2$. Let δ_0 be the minimal integer satisfying* (1)*. Then equality in* (1) *is attained for $\delta = \delta_0$ if and only if $p = s^2(k-1) - su$, where s is an odd, positive integer, and u is an integer of the same parity as k such that $0 \leq |u| \leq k-2$. Furthermore, $\delta_0 = (s-1)^2(k-1) - (s-1)u$.*

Proof. Writing p as in (25), a direct computation (as in Lemma 7.4) shows that equality in (1) is attained for $\delta = \delta_0$ only if $\delta_0 = (4m^2-1)(k-1) + (2m+1)t + \lambda + 1 + l$, with either $l = (t - 3k - \lambda + 2)/(2m+3)$ or $l = (k-2-\lambda-t)/(2m+1)$. One computes $p = (k-1)s^2 - \beta s$, with $s = 2m+3$ in the former case and $s = 2m+1$ in the latter, and β as defined in Remark 3.3. In particular, $0 \leq |\beta| \leq k-1$. Since p is odd, $s(k-1) - \beta$ must be odd, whence β has the same parity as k and $|\beta| \leq k-2$. The value of δ_0 is easily computed.

Conversely, if p and δ are of the given forms, one computes that equality is attained in (1). □

(As seen in the proof, $u = \beta$ from Remark 3.3.)

For $k = 2$, one has $\beta = 0$, whence the missing cases are $p = s^2$ with $s > 0$ odd and $\delta_0 = (s-1)^2$, i.e. $g = 2s-1$. In these cases, [FKP1, Prop. 7.7] shows, by completely different methods, the existence of a two-dimensional family of curves in $|H|$ with hyperelliptic normalizations, for $(S, H) \in \mathcal{K}_p$ general. However, it is not proved that these curves are nodal, so we cannot conclude that $V^2_{|H|,\delta_0} \neq \emptyset$.

8. The Mori cone of punctual Hilbert schemes and related conjectures

Finally we go back to the topic of § 2 (from which we keep the notation). Let $(S, H) \in \mathcal{K}_p$. For simplicity, we denote by $R_{p,\delta,k}$ the rational curves (or their classes) in $\mathrm{Hilb}^k(S)$ associated to the g^1_k on the normalizations of curves in $V^k_{|H|,\delta}$ satisfying (9) and (10). Theorems 0.1 and 3.1 determine the triples (p, k, δ) for which the rational curves $R_{p,\delta,k}$ exist, with the exception of the cases described in Proposition 7.5 for odd p. By Lemma 2.1, the class of $R_{p,\delta,k}$ in $N_1(\mathrm{Hilb}^k(S))$ is $H - (p - \delta + k - 1)\mathfrak{r}_k$. Given p and k, the curve $R_{p,\delta,k}$ that is the closest to the border of the Mori cone is the one corresponding to a minimal δ. We call such a curve (or class) *optimal*.

Remark 8.1. As $\rho(p,l,kl+\delta-1)=\rho(p,l,kl+\delta)-l-1$ for any positive integer l, the curve $R_{p,\delta,k}$ is optimal if $\rho<\alpha+1$ and not optimal if $\rho\geq\alpha+1+\epsilon$, where ρ is as in Remark 3.3, α as in (1) and $\epsilon=p-2\lfloor p/2\rfloor$. This follows from Theorems 3.1 and 0.1, respectively.

A result by Huybrechts [Hu2, Prop. 3.2] (resp. Boucksom [Bo]) says that a divisor D on a hyperkähler manifold X is nef (resp. ample) if and only if $q(D)>0$ and $D\cdot R\geq 0$ (resp. >0) for any (possibly singular) *rational* curve $R\subset X$. As a consequence, the Mori cone of X is generated by classes of rational curves. In view of this, Theorem 0.1 allows a small step towards the determination of the ample (or nef) cone:

Proposition 8.2. *Let $(S,H)\in\mathcal{K}_p$. If the $\mathbb{Q}$-divisor $D=H-t\mathfrak{e}_k$ in $\mathrm{Hilb}^k(S)$ is ample (resp. nef), then*

$$t<\frac{2(p-1)}{p-\delta_0+k-1}\quad\textit{(resp. }\leq\textit{)},\tag{28}$$

where δ_0 is the minimal integer satisfying (2).

Proof. We may assume that (S,H) is general. By Theorem 0.1, the class $R_{p,\delta_0,k}$ is effective, whence

$$D\cdot R_{p,\delta_0,k}=\Big(H-t\mathfrak{e}_k\Big)\cdot\Big(H-(p-\delta_0+k-1)\mathfrak{r}_k\Big)=2(p-1)-t(p-\delta_0+k-1)>0\text{ (resp. }\geq 0).$$

□

8.1. **Conjectures of Hassett and Tschinkel.** Hassett and Tschinkel conjecture in [HT1, Conj. 1.2] that, for any polarized variety $(X,\mathfrak{g})$ deformation equivalent to $\mathrm{Hilb}^k(S)$ with S a $K3$ surface, a 1-cycle R is effective if and only if $R\cdot\mathfrak{g}>0$ and $q(R)\geq-(k+3)/2$. The "only if" part in the case $k=2$ has been proved in [HT2]. They also conjecture in [HT1, Thesis 1.1] that $-(k+3)/2$ is the self-intersection of the lines in any $\mathbb{P}^k\subset X$. This latter conjecture has been verified for $k=2$ in [HT2], $k=3$ in [HHT] and $k=4$ in [BJ]. Other self-intersections have different geometrical properties from the point of view of birational geometry: primitive generators R of extremal rays on a hyperkähler manifold such that the associated extremal contraction is divisorial must satisfy $-2\leq q(R)<0$ by [HT1, Thm. 2.1].

As noted in § 3, our Corollary 3.4 verifies the first conjecture above for *any* rational curve from a g^1_k on a normalizations of a curve in $|H|$, where $(S,H)\in\mathcal{K}_p$ with $\mathrm{Pic}(S)\simeq\mathbb{Z}[H]$. For the curves $R_{p,\delta,k}$ obtained from Theorem 0.1, we have, by Corollary 3.4,

$$q(R_{p,\delta,k})=2(p-1)-\frac{(p-\delta+k-1)^2}{2(k-1)}=2(\rho-1)-\frac{\beta^2}{2(k-1)},\tag{29}$$

with ρ as in Remark 3.3 and α as in (1). (Recall that $\rho\geq 0$ and $-(k-1)<\beta\leq k-1$.) We can deduce the existence of the curves with lowest self-intersesction in Hassett-Tschinkel's conjecture:

Proposition 8.3. *Let $(S,H)\in\mathcal{K}_p$ be general and $R_{p,\delta,k}$ as above. Then $q(R_{p,\delta,k})=-\frac{k+3}{2}$ if and only if $p=s(s+1)(k-1)$ for an integer $s\geq 1$ and $R_{p,\delta,k}$ is optimal.*

Proof. By (29), we have $q(R_{p,\delta,k})=-\frac{k+3}{2}$ if and only if $\rho=0$ and $\beta=k-1$, in which case $R_{p,\delta,k}$ is optimal. A straightforward computation yields $g=2\alpha(k-1)$ and $p=\alpha(\alpha+1)(k-1)$. Conversely, if p is of the given form, one checks that $\delta=p-2s(k-1)$ verifies (2) and $q(R_{p,\delta,k})=-\frac{k+3}{2}$. □

If $k=2$, the self-intersection $-5/2$ is obtained if and only if $p=s(s+1)$ for an integer $s\geq 1$, which are exactly the genera covered in [FKP1, Props. 7.2], where curves with self-intersections $-5/2$ in $\mathrm{Hilb}^2(S)$ were obtained by completely different methods.

Since $N_1(\mathrm{Hilb}^k(S))$ has rank two for general $(S,H)\in\mathcal{K}_p$, we have:

Corollary 8.4. *Let $(S,H)\in\mathcal{K}_p$ be general such that $p=s(s+1)(k-1)$ for an integer $s\geq 1$. Assume that the "only if" part of* [HT1, Conj. 1.2] *holds true (e.g., $k=2$). Then the Mori cone of $\mathrm{Hilb}^k(S)$ is generated by $\mathfrak{r}_k$ and the optimal class $R_{p,\delta,k}$.*

For each k, the possible values of $q(R_{p,\delta,k})$ are a set of rationals of the form on the right hand side of (29). All such values are attained by an optimal curve:

Proposition 8.5. *Fix any integer $k \geq 2$. For any pair of integers (ρ, β) with $\rho \geq 0$ and $0 \leq \beta \leq k-1$, there are infinitely many positive integers p such that $\mathrm{Hilb}^k(S)$ for general $(S,H) \in \mathcal{K}_p$ contains a rational curve $R_{p,\delta,k}$ satisfying $q(R_{p,\delta,k}) = 2(\rho-1) - \frac{\beta^2}{2(k-1)}$.*

Proof. Pick, for instance, p as in (25) with $k-1 \leq t < 3(k-1)$ and let $\delta = \delta_0$ as in Lemma 7.4. One checks that this is also the minimal δ satisfying (1). Then $\beta = 2(k-1) - t$ and $\rho = \lambda + \epsilon$ and the result follows by choosing suitable λ, t and ϵ and varying $m \in \mathbb{Z}^+$ in (25), such that $\lambda \leq 2m$. □

In particular, this proposition may suggest a conceptual explanation of self-intersection numbers of extremal rays with negative self-intersection (cf. [HT1, beg. of § 4]). If $k = 2$, Proposition 8.5 gives the negative self-intersection numbers $-1/2$, -2 and $-5/2$ for optimal curves. This is in accordance with [HT3, Conj. 3.1], where Hassett and Tschinkel conjecture that the Mori cone should be generated by classes of curves R with positive intersection with some polarizing class and such that either $q(R) \geq 0$ or $q(R) = -1/2$, -2 or $-5/2$. (It was proved in [HT2] that the cone generated by these classes *contains* the Mori cone.) If $k = 3$ we obtain the negative self-intersection numbers -3, $-9/4$, -2, -1 and $-1/4$, which are the numbers in [HT1, Table H3], except for -1. In the case $k = 4$ we obtain $-7/2$, $-8/3$, $-13/6$, -2, $-3/2$, $-2/3$, $-1/6$, which are the numbers in [HT1, Table H4], except for $-3/2$. Thus, our results provide extensions of the examples in [HT1], both to all $k \geq 2$ and for infinitely many polarization genera p.

Taking into account Proposition 8.5, we are tempted to propose the following extension of [HT1, Conj. 1.2] and [HT3, Conj. 3.1]:

Conjecture 8.6. *The Mori cone of a polarized hyperkähler manifold $(X, \mathfrak{g})$ deformation equivalent to $\mathrm{Hilb}^k(S)$, with $(S,H) \in \mathcal{K}_p$, is generated by 1-cycles R such that $R \cdot \mathfrak{g} > 0$ and $q(R) \in \{2(\rho-1) - \frac{\beta^2}{2(k-1)} \mid \rho \geq 0,\ 0 \leq \beta \leq k-1\}$.*

In view of Corollary 8.4, it is also tempting to make the following:

Conjecture 8.7. *Let $(S,H) \in \mathcal{K}_p$ be general, with $p \geq 2$. The Mori cone of $\mathrm{Hilb}^k(S)$ is generated by $\mathfrak{r}_k$ and the optimal curve $R_{p,\delta,k}$.*

In the cases of Proposition 7.5, where we still have not proved the existence of the optimal curve, this conjecture must be taken with an additional warning.

A substantial step towards Conjecture 8.7 would be to extend our analysis to the nonprimitive case, i.e. to curves in $|mH|$ for $m > 1$. A consequence of Conjecture 8.7 would be that the bound (28) is optimal (possibly with δ_0 the minimal integer satisfying (1) in the cases of Proposition 7.5).

We will see in Proposition 8.8 below that Conjecture 8.7 holds in certain cases where $q(R_{p,\delta,k}) = 0$.

8.2. **Curves of self-intersection zero and conjectures of Huybrechts and Sawon.** The existence of curves (or divisors, cf. (6)) with Beauville-Bogomolov self-intersection zero on a hyperkähler manifold X is conjectured by Huybrechts [GHJ, § 21.4] and Sawon [Sa1, Conj. 4.2] (see also [Sa2, Conj. 1] to imply that X is birational to a Lagrangian fibration (Hassett and Tschinkel make in [HT3] the same conjecture in dimension 4). The existence of a nontrivial *nef* divisor D with $q(D) = 0$ is a *necessary* condition for a hyperkähler manifold to be a Lagrangian fibration (see e.g. [Sa1, § 4.1 and Rmk. 4.2]), and it is yet another conjecture of Sawon's [Sa2, Conj. 4.2] that this is also a *sufficient* condition. We also mention Matsushita's result [Ma] that any fibre space structure of a $2k$-dimensional projective hyperkähler manifold with projective base has the property that the general fibre is a k-dimensional Lagrangian subvariety that is abelian up to a finite unramified cover, and the base has the same Hodge numbers as $\mathbb{P}^k$ if smooth.

In the case $X = \mathrm{Hilb}^k(S)$ with $(S,H) \in \mathcal{K}_p$ such that $\mathrm{Pic}(S) \simeq \mathbb{Z}[H]$, a straightforward computation shows that a divisor $mH - n\mathfrak{e}_k$ is isotropic if and only if $m^2(p-1) = n^2(k-1)$ and this occurs if and only if $(p-1)(k-1)$ is a square (cf. [Sa2, § 1]). In the "primitive" case, that is, $m = 1$ and $p = n^2(k-1)+1$, Sawon proves that $\mathrm{Hilb}^k(S)$ is indeed a Lagrangian fibration [Sa2, Thm. 2]. As we will see in Corollary 8.11 below, this result cannot be generalized to all cases with $m > 1$.

A consequence of Sawon's result is that Conjecture 8.7 holds in certain numerical cases:

Proposition 8.8. *Let $(S,H) \in \mathcal{K}_p$ be general and $k \geq 2$ be any integer. Assume that $p = n^2(k-1)+1$ for some $n \geq 2$. Then $\mathfrak{r}_k$ and $R = R_{p,p-(2n-1)(k+1)+1,k}$ generate the Mori cone of $\mathrm{Hilb}^k(S)$.*

Proof. Let D be any isotropic divisor. Then, by [Sa2, Thm. 2], the class of D is proportional to the class of the fiber of the Lagrangian fibration of $\mathrm{Hilb}^k(S)$, whence is nef. One computes that $\delta = (2n-1)(k+1)+1$ satisfies (2) (and is also the minimal integer satisfying (1)), so that R exists, and $R \cdot D = 0$, so that R lies on the boundary of the Mori cone. $\square$

More generally, the $R_{p,\delta,k}$ with self-intersection zero always exist in the cases where an isotropic divisor exist:

Proposition 8.9. *Fix any integer $k \geq 2$. Let $(S,H) \in \mathcal{K}_p$ be general, with $p \geq 2$, and let $R_{p,\delta,k}$ be as above. Then there is a δ such that $q(R_{p,\delta,k}) = 0$ if and only if $(k-1)(p-1) = s^2$, for a positive integer s. In this case $\delta = p - 2s + k - 1$.*

Proof. By (29), one has $q(R_{p,\delta,k}) = 0$ if and only if $(k-1)(p-1) = s^2$ and $\delta = p - 2s + k - 1$. Conversely, such p and δ verify (2). $\square$

Corollary 8.10. *Let $(S,H) \in \mathcal{K}_p$ be general, with $p \geq 2$, and $k \geq 2$ be any integer. Assume that D is a nontrivial isotropic divisor in $\mathrm{Hilb}^k(S)$ (so that $(k-1)(p-1) = s^2$, with $s \geq 1$). Then the rational curve $R = R_{p,p-2s+k-1,k}$ satisfies $D \cdot R = 0$. In particular, if*

$$(k-1)(\alpha+1)^2 - (2s+1)(\alpha+1) + p \geq \epsilon, \quad \textit{where } \alpha := \Big\lfloor \frac{2s-k+1}{2(k-1)} \Big\rfloor \quad \textit{and } \epsilon = p - 2\Big\lfloor \frac{p}{2} \Big\rfloor, \tag{30}$$

then D is not nef, since the rational curve $R' = R_{p,p-2s+k-2,k}$ satisfies $D \cdot R' < 0$.

Proof. The condition (30) is equivalent to $\rho \geq \alpha + 1 + \epsilon$ and the result follows from Remark 8.1. $\square$

As a consequence, we obtain an additional necessary condition for $\mathrm{Hilb}^k(S)$ to be a Lagrangian fibration, showing that Sawon's result [Sa2, Thm. 2] cannot be generalized to all cases where a primitive divisor exists:

Corollary 8.11. *Let $(S,H) \in \mathcal{K}_p$, with $p \geq 2$, and $k \geq 2$ be any integer. If $\mathrm{Hilb}^k(S)$ is a Lagrangian fibration, then $(k-1)(p-1) = s^2$, with $s \geq 1$, and*

$$(k-1)(\alpha+1)^2 - (2s+1)(\alpha+1) + p < \epsilon, \quad \textit{where } \alpha := \Big\lfloor \frac{2s-k+1}{2(k-1)} \Big\rfloor \quad \textit{and } \epsilon = p - 2\Big\lfloor \frac{p}{2} \Big\rfloor. \tag{31}$$

Of course, $\mathrm{Hilb}^k(S)$ may still be *birational* to a Lagrangian fibration in the cases where (31) is not satisfied, in accordance with Huybrechts-Sawon's conjecture.

A consequence of Conjecture 8.7 would be that in the cases where the curve $R = R_{p,p-2s+k-1,k}$ from Proposition 8.9 and Corollary 8.10 is optimal, then any isotropic divisor D is nef. It is not known to us, cf. [HT3, Rem. 3.12], whether a nef isotropic divisor on a hyperkähler manifold is necessarily semiample, so that a multiple of it defines a morphism. Taking into account Conjecture 8.7 and Corollary 8.10, the following is a natural generalization of another conjecture of Hassett and Tschinkel [HT3, Conj. 3.8] and a refinement of Huybrechts-Sawon's ([GHJ, § 21.4],[Sa1, Conj. 4.2]) and Sawon's [Sa2, Conj. 4.2] conjectures:

Conjecture 8.12. *Let $(S,H) \in \mathcal{K}_p$ be general. Then $\mathrm{Hilb}^k(S)$ is a Lagrangian fibration if and only if $(k-1)(p-1) = s^2$, with $s \geq 1$, and*

$$(k-1)(\alpha+1)^2 - (2s+1)(\alpha+1) + p < 0, \quad \textit{where } \alpha := \Big\lfloor \frac{2s-k+1}{2(k-1)} \Big\rfloor, .$$

More precisely, the Lagrangian fibration is a Mori fibre space structure with respect to the extremal ray $R_{p,p-2s+k-1,k}$.

For even p, the "only if" part of this conjecture follows from Corollary 8.11.

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